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ORDINARY MEETING, JUNE 3, 1867.

CAPTAIN E. G. FISHBOURNE, R.N., C.B., IN THE CHAIR.

The Minutes of the last Meeting were read and confirmed, after which the following paper was read by the author:—

ON THE GEOMETRICAL ISOMORPHISM OF CRYSTALS AND THE DERIVATION OF ALL OTHER FORMS FROM THOSE OF THE CUBICAL SYSTEM. By REV. WALTER MITCHELL, M.A.

1. WHEN elementary substances, or their chemical combinations, pass from a state of vapour; or from a fluid condition into that of a solid; or if they are deposited by evaporation from a fluid holding them in solution, there is a tendency of their particles to arrange themselves according to certain laws of symmetry.

2. Thus solids more or less symmetrical, and with few exceptions bounded by smooth, plane, or flat surfaces, are produced. Such solids are called *crystals*, and their plane surfaces are termed *faces*.

3. Some crystals are remarkable for perfect symmetry of form. Among these may be found solids formed with mathematical accuracy, whose geometrical properties had fascinated the ancient geometers ages before they were known to exist in the productions of nature. Others are exceedingly complex, being formed by the combination of faces parallel to those belonging to several simpler forms; the relative positions of these simpler forms to each other being regulated by certain mathematical laws.

4. The more complex forms being reduced to the combination of the simplest from which they can be derived, it is found that all the simpler forms can be grouped together in six distinct classes or systems.

5. The crystals of any one substance may generally be reduced to forms belonging to one system; but there seems to be no limit to the number of combinations of different species of these forms which may take place in any individual crystal.

6. To the rule that all the crystals of a particular substance should have their faces parallel to those of the forms of one system, there are numerous exceptions.

7. The following are the six systems :—

1st. The *Cubical;* called also the tesseral, tessular, octahedral, regular, isometric, and monometric.

2nd. The *Pyramidal;* called also the tetragonal, square prismatic, quadratic, monodimetric, dimetric, four-membered, viergliedrig, and the two-and-one axial.

3rd. The *Rhombohedral;* called also the hexagonal, monotrimetrical, sechsgliedrig, and the three-and-one axial.

4th. The *Prismatic;* called also the rhombic, trimetric, binary, unisometric, orthotype, orthorhombic, zweigliedrig, and one-and-one axial.

5th. The *Oblique;* called also the monoclinohedric, hemiprismatic, hemiorthotype, clinorhombic, hemi-hedric-rhombic, augitic, zwei-und-eingliedrig, and the two-and-one-membered.

6th. The *Anorthic;* called also the doubly oblique, triclinic, triclinohedric, anorthotype, clinorhomboidal, tetarto-prismatic, tetarto-rhombic, eingliedrig, and the one-and-one-membered.

Cubical System.

8. The forms of the cubical system possess the highest possible degree of symmetry when compared with those of the other systems. They are divided into two groups,—the *holohedral,* or perfectly symmetrical, and the *hemihedral,* or half-symmetrical; the latter being derived from the former by being parallel to, or possessing only half their faces, grouped together after certain laws.

9. The *holohedral,* or perfectly symmetrical forms, are seven in number, and are shown on Plate I. Of these, three—the cube (fig. 1), the octahedron (fig. 7), and the rhombic dodecahedron (fig. 8), are invariable forms, each having but one species, and each the same invariable angles, either of their faces or inclination of their faces.

The remaining four forms are not invariable, and there are an infinite variety of species, each differing from the other in the angles of their faces and their inclinations to each other.

The half-symmetrical, or hemihedral forms, are represented in figs. 15, 17, 19, 21, 23, and 25, Plate III.

Holohedral forms, cubical system.

10. The Cube (fig. 1, Plate I.) is bounded by six equal faces, each face, such as $O_1O_5O_8O_4$, being a perfect square;

it has therefore eight solid angles, O_1, O_2, &c., O_8, each angle being formed by the union of three planes; and twelve equal edges, such as O_1O_2, O_2O_3, &c. The inclination of any face to another is measured by the angle contained between two perpendiculars drawn from any point in the edge made by the intersection of the two faces, each on one of the adjacent faces. In the cube this inclination of two adjacent faces is 90°. The facial angles, or the angles between two edges of a face, such as $O_4O_1O_5$, are always 90°.

11. The OCTAHEDRON (fig. 7, Plate I.) is bounded by eight equal faces, each face, such as $C_1C_2C_3$, shown on a plane surface (fig. 33, Plate IV.), being an equilateral triangle. It has six solid angles, C_1, C_2, &c., C_6, each formed by the union of four planes, and twelve equal edges; the inclination of adjacent faces is an angle of 109° 28′, and the facial angle, such as $C_1C_2C_3$, is 60°.

12. The RHOMBIC DODECAHEDRON (fig. 5, Plate I.) is bounded by twelve equal faces; each face, such as $o_1C_2o_5C_3$ (fig. 30, Plate IV.), is a geometrical rhomb bounded by four equal lines, o_1C_2 being parallel to o_5C_3, and o_1C_3 to o_5C_2. The greater angles of the rhomb $C_2o_1C_3$ and $C_3o_5C_2$ being 109° 28′, and the lesser, $o_1C_2o_5$ and $o_1C_3o_5$, 70° 32′. It has twenty-four equal edges, such as C_1o_1, C_1o_2, &c., eight solid angles, o_1, o_2, &c., o_8, formed by the union of three planes, and six solid angles, C_1, C_2, &c., C_6, formed by the union of four planes. The inclination of adjacent faces is 120°. This form is called by some German writers the *granatoëdron*, as being a characteristic form of the *garnet*.

13. These three forms, the *cube*, *octahedron*, and *rhombic dodecahedron*, are called invariable forms, as, though differing in size, they always have similar faces and angles; that of the cube being a square, that of the octahedron an equilateral triangle, and that of the rhombic dodecahedron a rhomb whose larger angle is 109° 28′.

14. The four other forms (figs. 2, 3, 4, and 6, Plate I.) are called variable, each presenting an infinite variety of species, differing from each other in their angles of inclination and those of their faces.

15. The THREE-FACED OCTAHEDRON (fig. 6, Plate I.) is bounded by 24 equal faces, each being an isosceles triangle, $o_1C_2C_3$ (fig. 32, Plate IV.). These faces are so grouped together as to form a solid having eight solid angles, formed by the union of three planes, o_1, o_2, o_3, &c., o_8 (fig. 6); the plane angles being the largest of the isosceles triangles; and six solid angles, C_1, C_2, &c., C_6, each formed by the union of eight of the equal angles of the isosceles triangles.

There are 12 longer edges, such as C_1C_2, C_1C_3, &c., and 24 shorter, such as o_1C_1, o_1C_2, &c. The 12 longer edges are the edges of an octahedron. It may be formed by placing on every face of the octahedron a three-faced pyramid on a equilateral triangular base. The angles of these isosceles triangles differ in different species of the three-faced octahedron, within certain limits to be described hereafter.

The synonyms for this form are the *pyramidal octahedron*, *triakisoctahedron*, *trioctahedron*, and *galenoid*.

16. The FOUR-FACED CUBE (fig. 2, Plate I.) is bounded like the last by 24 equal faces, each being an isosceles triangle, such as $C_1o_1o_4$ (fig. 34, Plate IV.), but grouped so together as to form a solid having six solid angles, C_1, C_2, &c., C_6 (fig. 2), each formed by the union of four of the largest angles of the isoscles triangles, and eight solid angles, o_1, o_2, &c., o_8 (fig. 2), formed by the union of six of the equal angles of the isosceles triangles. This form has 24 shorter edges, such as C_1o_1, C_1o_2, &c., and 12 longer ones, such as o_1o_4, o_1o_5, &c. The 12 longer edges are those of a cube.

It may be formed by placing on every face of the cube a four-faced pyramid on a square base.

The angles of the isosceles triangles differ for each particular species of the *four-faced cube*.

Synonyms.—*Pyramidal cube*, *hexatetrahedron*, *tetrakis-hexahedron*, and *fluoride*.

17. The TWENTY-FOUR-FACED TRAPEZOHEDRON (fig. 4, Plate I.) is bounded by 24 equal faces, each face being a deltoid or trapezium, $C_1d_1o_1d_2$ (fig. 39, Plate IV.); that is, a four-faced figure having two longer equal sides, C_1d_1 and C_1d_2, and two shorter equal sides, o_1d_2, o_1d_1. These 24 equal trapeziums are so grouped together as to form a solid having six solid angles, C_1, C_2, &c., C_6, formed by the union of the plane angles of four trapeziums, equal to $d_1C_1d_2$; eight solid angles, o_1, o_2, &c., o_8, formed by the union of the plane angles of three trapeziums, equal to $d_1o_1d_2$; and 12 solid angles, d_1, d_2, &c., d_{12}, formed by the union of the plane angles of four trapeziums, equal to $C_1d_1o_1$. This form has 24 equal longer edges, such as C_1d_1, C_1d_2, and 24 shorter edges, such as o_1d_1, o_1d_2, &c. The angles of the deltoids or trapeziums differ for each particular species of the *twenty-four-faced trapezium*.

Synonyms.—*Icositessarahedron*, *icositetrahedron*, *trapezohedron*, and *leucitoid*.

18. The SIX-FACED OCTAHEDRON (fig. 3, Plate I.) is bounded by 48 equal faces, each face being a scalene triangle, $C_1o_1d_2$ (fig. 36, Plate IV.). These 24 triangular faces are so grouped together as to form a solid having six solid angles, C_1, C_2, &c.,

C_6, each formed by the union of eight equal plane angles at the points C_1, C_2, &c.; eight solid angles, formed by the union of six equal plane angles at the points o_1, o_2, &c., o_8; and 12 solid angles, formed by the union of four plane angles at the points d_1, d_2, &c., d_{12}.

This form has 24 edges, each equal to the edge C_1d_1, 24 each equal to the edge C_1o_1, and 24 each equal to o_1d_1.

The angles of the triangular faces of this form differ for each particular species of *the six-faced octahedron.*

Synonyms.—*Hexakis-octahedron, hexoctahedron, tetrakonta-oktaëdron, pyramidal granatohedron, triagonal polyhedron,* and *adamantoid.*

19. These seven forms, grouped together on Plate I., have this relation in nature, that any substance forming crystals of any one of these forms may, and does sometimes, form crystals of any one of the other forms, or parallel to their faces. But when these forms are combined on any one crystal, as in fig. 29*, Plate IV.*, the forms to which the faces are parallel, except in the case of what are called twin crystals, always have a certain fixed position with regard to each other. These forms have not only this natural relationship to each other, but they have also certain geometrical relations, which we shall proceed to describe.

20. Looking at Plate I., the forms present no relationship to each other. Plate II. shows them conuected together by beautiful geometrical laws.

21. In Plate II. we see that each of the six other forms can every one of them be inscribed, as geometers term it, in the cube.

Fig. 8, Plate II., shows the *cube* having each of its faces divided into eight equal triangles, by joining the opposite angles of each square by two diagonals, such as O_1O_8, O_4O_5, meeting in C_2, the centre of the face, and by two other lines, such as D_1D_9, D_8D_5, also meeting in C_1, and joining the centres D_1, D_9 of the edges O_1O_4, O_5O_8, and D_5, D_8, the centres of the edges O_1O_5 and O_4O_8.

Fig. 9, Plate II., shows the *Four-faced cube* inscribed in the cube, and we see that the six solid angles of the twenty-four faced cube, C_1, C_2, &c., C_6 touch the six centres of the six faces of the circumscribing cube.

Fig. 10. The *Six-faced octahedron* inscribed in the cube, six of its solid angles, C_1, C_2, &c., C_6, touching the centres of the six faces of the circumscribing cube.

Fig. 11. The *Twenty-four-faced trapezohedron* inscribed in the cube, six of its solid angles, C_1, C_2, &c., C_6, touching the centres of the six faces of the circumscribing cube.

Fig. 12. The *Rhombic dodecahedron* inscribed in the cube,

six of its solid angles, C_1, C_2, &c., C_6, touching the centres of the six faces of the circumscribing cube.

Fig. 13. The *Three-faced octahedron* inscribed in the cube, six of its solid angles, C_1, C_2, &c., C_6, touching the centres of the six faces of the circumscribing cube.

Fig. 14. The *Octahedron* inscribed in the cube, its six solid angles C_1, C_2, &c., C_6, touching the centres of the six faces of the circumscribing cube.

Cubical Axes.

22. The lines formed by joining the opposite centres of the faces of the cube C_1C_6, C_5C_3, and C_2C_4 (fig. 27, Plate IV.), are called the *cubical axes* of the cube. These three lines are equal to each other, and are perpendicular each to two opposite faces of the cube; they intersect in A, the centre of the cube. In fig. 27 two other sets of axes are shown, four O_1O_7, O_2O_8, O_3O_5, and O_4O_6, joining the opposite solid angles O_1, O_2, &c., O_6, of the cube; six others, D_1D_{11}, D_2D_{12}, D_3D_9, &c., D_8D_6, joining the opposite centres D_1, D_2, &c., D_{12} of the edges of the cube; both sets of axes passing through A, the centre of the cube. The four axes O_1O_7, &c., O_4O_8, fig. 27, Plate IV., are evidently the four diagonals of the cube, and are represented fig. 9, fig. 10, &c., to fig. 14, Plate II., by lines marked thus —· —· —· The line D_1D_{11}, fig. 27, is parallel and equal to a line drawn from O_1 to O_6, and is therefore equal to a diagonal of one of the faces of the cube. The 12 axes D_1D_{11}, D_2D_{12}, &c., D_6D_8, are therefore each equal to a diagonal of the face of the cube. These lines are thus represented — — — —, fig. 9, fig. 10 to fig. 14, Plate II.

Octahedral Axes.

23. If the equilateral triangle $C_1C_2C_3$, representing one of the faces of the octahedron (fig. 33, Plate IV.), has its three sides bisected by d_1, d_2, d_5, and C_1d_5, C_2d_2, and C_3d_1 be drawn meeting each other in the point o_1, this point o_1 will represent the centre of gravity of the triangle $C_1C_2C_3$, and any of the shorter lines do will be a third of the longer one, Cd. The octahedron inscribed in the cube fig. 14, Plate II., has all its edges bisected by the points d_1, d_2, &c., d_{12}, and each equilateral triangle divided into six triangles by lines Cd meeting in o_1, o_2, &c., o_8, the centres of the eight faces of the octahedron.

It will be seen in fig. 14 that the six axes, such as D_2D_{12}, pass through two opposite bisections, d_2, d_{12}, of the opposite edges C_1C_3 and C_5C_6 of the octahedron.

The four axes, such as O_1O_7, pass through the centres o_1, o_7 of the opposite and parallel faces, $C_1C_2C_3$ and $C_6C_5C_4$ of the octahedron, and are perpendicular to both of them.

Owing to this property, the four axes O_1O_7, &c., O_4O_6, are called the *octahedral* axes of the cube.

24. This property may be demonstrated as follows :—

Describe a square (fig. 27*, Plate IV.*), $AC_1D_1C_2$, having each of its sides = O_1D_5 (fig. 27, Plate IV.).

$AC_1D_1C_2$ is evidently a fourth of the square $O_1O_5O_8O_4$, forming a face of the cube (fig. 27, Plate IV.).

Draw the diagonals of the square C_1C_2, and AD_1, meeting in the point d_1. C_1C_2 bisected in d_1 will represent on a plane surface in (fig. 27*, Plate IV.*) the edge of the octahedron $C_1d_1C_2$ seen in perspective in (fig. 14, Plate II.).

Produce D_1C_1 and C_2A (fig. 27*) to O_1 and D_5, making C_1O_1 and AD_5 each = AD, a diagonal of the square $D_1C_1AC_2$.

Join O_1D_5, make $Ad_5 = Ad_1$. Join C_1d_5 and AO_1, meeting in o_1.

Then $C_1o_1d_5$ and Ao_1O_1 (fig. 27*) represent on a plane surface the lines similarly shown in perspective in (fig. 14, Plate II.)

25. To facilitate calculation we shall choose one of the sides of the square $C_1AC_2D_1$ as our unit.

Then $AD_1 = \sqrt{2}$ and $Ad_1 = Ad_5 = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

By plane trigonometry $\tan Ad_5C_1 = \dfrac{AC_1}{AD_5} = \dfrac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$.

And angle $Ad_5C_1 = 54° \; 44' \; 8''$.

Now (fig. 14, Plate II.) the lines C_1d_5 and C_6d_5 are both by construction perpendicular to the edge C_2C_3 of the octahedron of two adjacent faces at the point d_5.

The angle $C_1d_5C_6$ therefore measures the inclination of these faces; but this angle is evidently twice the angle Ad_5C_1 (fig. 27*, Plate IV.*). What is true with regard to the angle of inclination over the edge C_1C_6 is true by similarity and symmetry of construction of all the other edges of the octahedron. And therefore the angle of inclination of any two adjacent faces of the octahedron is $109° \; 28' \; 16''$.

26. Again (fig. 27*, Plate IV.*) $\tan AO_1D_5 = \dfrac{AD_5}{O_1D_5} = \sqrt{2}$.

but $\tan Ad_5C_1 = \sqrt{2}$. Therefore $AO_1D_5 = Ad_5C_1$;

also $O_1AD_5 = 90° - Ao_1D_5 = 90 - Ad_5C_1$; consequently

$Ao_1d_5 = 90°$, and the line Ao_1 is perpendicular to C_1d_5 at the point o_1.

By symmetry of construction the line O_1o_1 (fig. 14, Plate II.)

is perpendicular to the three lines C_1d_5, C_2d_2, and C_3d_1, and consequently to the plane face $C_2C_2C_3$ of the octahedron.

Likewise by symmetry of construction each of the four axes O_1O_7, &c., O_4O_6, are respectively perpendicular to two opposite and parallel faces of the octahedron.

27. From triangle AO_1D_5 (fig. 27*) we have

$$AO_1^2 = O_1D_5^2 + AD_5^2 = 1 + 2 = 3.$$

Therefore $AO_1 = \sqrt{3}$.

In right-angled triangle C_1Ad_5; $C_1d_5^2 = C_1A^2 + Ad_5^2 = 1 + \frac{1}{2} = \frac{3}{2}$

Therefore $C_1d_5 = \sqrt{\frac{3}{2}}$.

But triangles Ao_1d_5 and AD_5O_1 are similar.

Therefore $\frac{Ao_1}{Ad_5} = \frac{AD_5}{AO_1}$ and $Ao_1 = \frac{Ad_5 \cdot AD_5}{AO_1} = \frac{\sqrt{2} \cdot \frac{\sqrt{2}}{2}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Consequently $Ao_1 = \frac{1}{3}AO_1$.

Again by similar triangles Ad_5o_1 and C_1d_5A.

$\frac{o_1d_5}{d_5A} = \frac{Ad_5}{d_5C_1}$ $o_1d_5 = \frac{(Ad_5)^2}{d_5C_1} = \frac{1}{3}\sqrt{\frac{3}{2}} = \frac{1}{3}C_1d_3$. Also $Ad_5 = \frac{1}{2}AD_5$.

28. Hence, referring to (fig. 14, Plate II.), we see that when the octahedron is inscribed in the cube, the three cubical axes, C_1C_6, C_2C_4, and C_3C_5 join together the opposite solid angles of the octahedron. The four octahedral axes O_1O_7, O_2O_8, &c., O_4O_6, pass through the centres of two opposite faces of the octahedron and are perpendicular to them.

The points o_1, o_2, &c., being one-third of the distance of the centre of the cube from the solid angles O_1, O_2, &c., of the circumscribing cube.

Also that the six axes D_1D_{11}, &c., D_6D_8, joining the opposite centres of the edges of the cube, pass each through two opposite edges of the inscribed octahedron. The distance of the centre of the cube from the centre of the edge of the octahedron being half the distance of the centre of the edge of the cube from that point.

29. Referring to fig. 27*, Plate IV.*, we have already shown, § 25, that the angle Ad_5C_1 = angle $AO_1D_5 = 54° \ 44' \ 8''$, consequently, since $C_1AD_1O_1$ is by construction a parallelogram,

The angle $C_1AO_1 = 54° \ 44' \ 8''$, and the angle $O_1AD_5 = 35° \ 15' \ 52''$.

Hence the angle such as C_1AO_1 which any octahedral axis AO makes with any adjacent cubical axis AC is $54° \ 44' \ 8''$; and the angle such as O_1AD_5 which the octahedral axis OA makes with any adjacent axis AD_5 is $35° \ 15' \ 52''$. This latter axis is called a rhombic axis.

Rhombic Axes.

30. Describe a square $D_1C_1AC_2$ (fig. 28*, Plate IV.*) having its equal sides one-half the side or edge of the circumscribing cube. Join the diagonals C_1C_2 and D_1A meeting in d_1. Produce D_1C_1 to O_1 and C_2A to D_5, making C_1O_1 and AD_5 each $=AD_1$. Join C_1D_5 and O_1A meeting in o_1. Draw O_1d_5 perpendicular to AD_5. Then since $C_1O_1D_5A$ is a rectangular parallelogram, it follows AO_1 is bisected in o_1, $o_1d_5=\frac{1}{2}O_1D_5$ and $Ad_5=\frac{1}{2}AD_5$.

Then referring to (fig. 12, Plate II.),—the square $C_1D_1C_2A$ represents on a plane surface (fig. 28*), and the parallelogram $C_1AD_5O_1$ the same figures shown in perspective in (fig. 12, Plate II.); the former being one-fourth of a section of the cube drawn through the points $D_1D_3D_{11}D_9$, and the latter one-fourth of the section drawn through $O_3O_1O_5O_7$.

$C_1d_1C_2$, C_1o_1, o_1d_5, &c., representing the lines similarly marked in the perspective figure of the rhombic dodecahedron inscribed in the cube.

31. Now fig. 30, Plate IV. Draw $C_2C_3=C_1C_2$ (fig. 28*), on both sides C_2C_3 as base, describe two isosceles triangles having their equal sides, such as $C_2o_1=C_1o_1$ (fig. 28*); join the diagonals C_2C_3 and o_1o_5 meeting in d_5. $C_2o_5C_3o_1$ will represent on a plane surface a face of the rhombic dodecahedron, which can be inscribed in a cube whose edge is double C_2D_1 or O_1D_5 (fig. 27*).

32. (Fig. 28*, Plate IV.*) D_1d_1 is perpendicular to $C_1d_1C_2$, and also D_5d_5 is perpendicular to o_1d_5. Hence, referring to (fig. 12, Plate II.), D_1d_1 is perpendicular to $C_1d_1C_2$, and D_5d_5 is perpendicular to o_1d_5. Hence, by symmetry and similarity of construction, D_5d_5 is perpendicular to o_1o_5, and C_2C_3 meeting in d_5; and therefore D_5d_5 is perpendicular to the face $o_1C_2o_5C_3$ of the rhombic dodecahedron, and passes through d_5, its centre of gravity.

33. Hence by symmetry and similarity of construction comparing (fig. 12, Plate IV.) with (fig. 5, Plate I.), every axis D_1D_{11}, D_2D_{12}, D_3D_9, &c., D_6D_8, joining the opposite centres of the edges of the circumscribing cube, are each perpendicular to, and pass through the centres of gravity of opposite and parallel faces of the inscribed rhombic dodecahedron. Thus D_1D_{11} is perpendicular to $C_1o_1C_2o_4$, and $C_4o_6C_6o_7$, D_2D_{12} is perpendicular to $C_1o_1C_3o_2$ and $C_5o_8C_6o_7$, &c. From this property these axes are called the rhombic axes.

34. Again referring to (fig. 28*, Plate IV.*), we see that $Ao_1=\frac{1}{2}AO_1$ and $Ad_1=\frac{1}{2}AD_1$. Hence by similarity and symmetry of construction (fig. 12, Plate II.) we see that the rhombic dodecahedron, inscribed in the cube, touches the centre of

each face of the cube, C_1, C_2, &c., C_6, by one of its four-faced solid angles; cuts each octahedral axis AO_1, AO_2, &c., by o_1, o_2, &c., one of its three-faced solid angles, at a distance Ao_1 the $\frac{1}{2}$ of AO_1. Also each semi-rhombic axis cuts the centre of the rhombic face, such as $C_2o_1C_3o_5$ at d_5, Ad_5 being $\frac{1}{2}AD_5$.

To inscribe the three-faced Octahedron in the Cube.

35. (Fig. 29, Plate IV.) Describe the square $C_1D_1C_2A$, having each of its sides equal to O_1D_1, fig. 27. Draw the diagonals C_1C_2 and D_1A meeting in d_1.

Produce D_1C_1 and C_2A to O_1 and D_5, make AD_5 and C_1O_1 each equal to AD_1. Join O_1D_5. In AD_5 take $Ad_5=Ad_1$.

Produce AC_1 to M. For distance AM see § 37. Join d_5M, cutting AO_1 in o_1. Then join C_1o_1.

Then referring to (fig. 13, Plate II.), $C_1d_1C_2$ represents the edge of the three-faced octahedron, C_1o_1 and o_1d_5 the corresponding lines shown in perspective.

36. To draw the three-faced octahedron inscribed in the cube (fig. 27, Plate IV.).

Describe a square $O_1O_5O_8O_4$; draw O_4O_3 at such an angle and such a length that none of the edges or axes of the cube may obscure each other. Then draw O_1O_2, O_5O_6, and O_8O_7 parallel and equal to O_4O_3. Join O_3O_2, O_2O_6, O_6O_7, and O_7O_3. Also join O_1O_7, O_2O_8, O_3O_5, and O_4O_6 meeting in A, the centre of the cube. These diagonals of the cube are the four octahedral axes of the cube.

Bisect O_1O_2 in D_1, O_1O_2 in D_2, &c., O_8O_7 in D_{12}; join D_1D_{11}, D_2D_{12}, D_3D_9, D_4D_{10}, D_5D_7, and D_6D_8, all intersecting in A. These are the six rhombic axes of the cube.

Lastly take C_1 the intersection of the diagonals of the face $O_1O_2O_3O_4$, C_2 that of the diagonals of the face $O_1O_5O_8O_4$, &c. Join C_1C_6, C_2C_4, and C_3C_5 intersecting in A. These are the three cubical axes of the cube.

Then take a pair of proportional compasses and set them so that Ao_1 (fig. 29, Plate IV.) be the distance between the shorter legs, and AO_1 between the longer legs of the compass.

Then in fig. 27, take the distance AO_1 with the longer legs and mark off Ao_1 with the shorter; in the same way mark off the points o_2, o_3, &c., o_8, on the other octahedral axes.

Lastly (fig. 13, Plate II.) prick off from this construction of (fig. 27, Plate IV.) the points C_1, C_2, &c., C_6; D_1, D_2, &c., D_{12}; O_1, O_2, &c., O_8; and o_1, o_2, &c., o_8. Draw the same lines as in fig. 27.

Join C_1C_2, C_2C_3, &c., C_1o_1, C_2o_1, C_3o_1, C_1o_4, C_2o_4, C_5o_4, &c. Then d_1, d_2, &c., will be the points where the rhombic axes bisect the edges C_1C_2, C_1C_3, &c. Join with dotted lines d_1o_1, d_2o_1, &c.; then (fig. 13, Plate II.) will represent in perspective the three-faced octahedron inscribed in the cube.

In the solid itself the eight lines Oo are each equal O_1o_1 (fig. 29, Plate IV.), the twelve lines Dd are each equal D_1d_1, or D_5d_5 (fig. 29).

37. The distance of the point M from A (fig. 29, Plate IV.) is arbitrary, so long as AM is greater than AC_1.

For every point chosen for M, we have a value for Ao_1, which gives a distinct species of three-faced octahedron.

Speaking generally, taking AC_1 as a unit, AM may represent any whole number or fraction greater than unity.

The following values of AM have been observed in natural crystals:—

$AM = 2AC_1$, $\frac{3}{2}AC_1$, $4AC_1$, $\frac{7}{4}AC_1$, $\frac{5}{4}AC_1$, and $\frac{65}{64}AC_1$.

38. Comparing (fig. 29, Plate IV.) with (fig. 27*, Plate IV.*), we see that M coincides with C_1, and $Ao_1 = \frac{AO_1}{3}$ for the octahedron; and with (Plate IV.*, fig. 28*), $Ao_1 = \frac{AO_1}{2}$ and o_1d_5 is parallel to AC_1 in the rhombic dodecahedron. In which case the point M is said to be at an infinite distance from A.

39. Hence referring to figs. 12, 13, and 14, Plate II., we see that the point o_1 of the three-faced octahedron cuts the octahedral axis at some point between $\frac{AO_1}{2}$ and $\frac{AO_1}{3}$; there being a distinct species of three-faced octahedron for every one of these points; the distance Ao_1, Ao_2, and Ao_8 being the same for the same species.

40. Hence the rhombic dodecahedron, fig. 12, and the octahedron, fig. 14, are the two limiting forms of the three-faced octahedron.

41. If we construct (fig. 14) the edges of the cube in wire and all the lines of the octahedron, such as C_1d_5, C_3d_1, &c., in elastic threads; then if strings be fastened to o_1 tying together C_3d_1, C_2d_2, &c., and these strings pass over pulleys at the points o_1, o_2, &c., o_8, if they be pulled uniformly so that o_1, o_2, &c., o_8 pass from $\frac{AO_1}{3}$ to $\frac{AO_1}{2}$ along the octahedral axes, the model will show in that finite space of time every one of the infinite number of species of three-faced octahedrons that can theoretically lie between fig. 14, the octahedron, and fig. 12, the rhombic dodecahedron inscribed in the cube.

Looking at the three figures, 12, 13, and 14, we see that the twelve lines, such as $C_1d_1C_2$, the edges of the octahedron, remain unaltered, the changing lines being represented by C_1o_1 and o_1d_5. As the point o_1 travels from $\frac{AO_1}{3}$, fig. 14, to $\frac{AO_1}{2}$, fig. 12, the apex o_1 rises from the triangular base $C_1C_2C_3$, in fig. 14, till two adjacent planes, fig. 12, over the edge $C_1d_1C_2$, such as $o_1C_1C_2$ and $o_4C_1C_2$, fig. 13, come into the same plane, fig. 12.

Fig. 14 having eight plane faces, passes through an infinite series of forms, such as fig. 13, bounded by 24 plane faces, and terminates fig. 12 in a form bounded by twelve plane faces.

42. If (fig. 32, Plate IV.) we draw $C_2C_3 = C_2C_1$ (fig. 29, Plate IV.), and describe on C_2C_3 the isosceles triangle $C_2o_1C_3$, having each of its equal sides C_2o_1 and $C_3o_1 = C_1o_1$ (fig. 29), then the triangle $C_2o_1C_3$ will represent, on a plane surface, one of the 24 equal faces of the three-faced octahedron which can be inscribed in a cube whose face is equal $O_1O_4O_8O_5$, fig. 27.

43. Twenty-four of these triangles drawn on a plane surface of cardboard can be cut out and folded together so as to make a model of the three-faced octahedron. Such drawings are called "nets." Nets ready drawn and fit for cutting and folding and making models for all the principal forms of crystals, by Mr. James B. Jordan, are published in Murby's Science and Art department Text Book, "Elementary Crystallography."

44. Referring to (Plate IV., fig. 29), we see that it is the distance of the point M from A which determines the point o_1 in AO_1; or referring to (fig. 13, Plate II.) the eight points o_1, o_2, &c., o_8, which taken at equal distances from the centre of the circumscribing cube in the octahedral axes, determine the species of the three-faced octahedron. If (fig. 29, Plate IV.) we take AC_1 as unity and call $AM = m$, m then determines the species of the three-faced octahedron, m being any whole number or fraction greater than unity.

45. Now comparing (fig. 29, Plate IV.) with (fig. 13, Plate II.) we see that any particular face, such as $o_1C_2C_3$, cuts two cubical axes AC_2 and AC_3 in points C_2 and C_3, and the third axis AC_1 produced in M, or at distances AC_2, AC_3, and AM; or 1, 1, and m. Since the line o_1d_5 cuts AC_1 in M, consequently the plane $o_1C_2C_3$ produced also cuts AC_1 in M. What is true for one face, by the similarity and symmetry of construction of the three-faced octahedron (fig. 13, Plate II.), is true for every other of the 24 faces. If m be a fraction represented by $\frac{h}{k}$, then the following are the most received symbols for the three-faced octahedron.

$\frac{h}{k}O$ Naumann; $k\,h\,h$ Miller; and $a^{\frac{k}{h}}$ Brooke, Levy, and Des Cloizeau.

46. The following species have been observed in nature, having these respective values for m; viz., 2, 3, $\frac{5}{2}$, 4, $\frac{7}{4}$, $\frac{5}{4}$, and $\frac{65}{64}$. The annexed table gives the respective symbols of the

principal crystallographers for these forms, together with the minerals in which faces of them have been found.

Naumann.	Miller.	Brooke,&c.	Minerals.
$2O$	1 2 2	$a^{\frac{1}{2}}$	Amalgam. Fluor. Pharmaco- Argentite. Franklinite. siderite. Blende. Galena. Pyrite. Cuprite. Magnetite. Skutterudite. Diamond. Perowskite. Spinelle.
$3O$	1 3 3	$a^{\frac{1}{3}}$	Cuprite. Fluor. Galena.
$\frac{3}{2}O$	2 3 3	$a^{\frac{2}{3}}$	Fahlerz. Garnet. Cuprite.
$4O$	1 4 4	$a^{\frac{1}{4}}$	Galena. Kerate.
$\frac{7}{4}O$	4 7 7	$a^{\frac{4}{7}}$	Galena.
$\frac{5}{4}O$	4 5 5	$a^{\frac{4}{5}}$	Galena.
$\frac{65}{64}O$	64 65 65	$a^{\frac{64}{65}}$	Alum.

47. To find the ratio of the octahedral axis of the three-faced octahedron to that of the circumscribing cube, or of Ao_1 to AO_1.

Fig. 29, Plate IV. By construction $O_1D_5=1$ and $AD_5=\sqrt{2}$.

Therefore $\tan AO_1D_5=\sqrt{2}=54^\circ\ 44'$;

And therefore $O_1AD_5=35^\circ\ 16'$.

Also $\tan Md_5A=\dfrac{AM}{Ad_5}=\dfrac{m}{\frac{1}{2}\sqrt{2}}=m\sqrt{2}$.

But $Ao_1d_5=180-(o_1Ad_5+Ad_5M)=180^\circ-35^\circ\ 16'-Ad_5M$.
$=144^\circ\ 44'-Ad_5M$.

Hence $\sin Ao_1d_5=\cos(90-Ao_1d_5)=\cos(90-144^\circ 44'+Ad_5M)$.
$=\cos\ (Ad_5M-54^\circ\ 44')$.

But in triangle Ao_1d_5, $\dfrac{Ao_1}{Ad_5}=\dfrac{\sin Ad_5M}{\sin Ao_1d_5}$

Therefore

$$Ao_1=Ad_5\frac{\sin Ad_5M}{\sin Ao_1d_5}=Ad_5\frac{\sin Ad_5M}{\cos\ (Ad_5M-54^\circ\ 44')}$$

$$=Ad_5\frac{\sin Ad_5M}{\cos Ad_5M\cos 54^\circ\ 44'+\sin Ad_5M\sin 54^\circ\ 44'}$$

$$=Ad_5\frac{\tan Ad_5M}{\cos 54^\circ\ 44'+\tan Ad_5M\sin 54^\circ\ 44'}$$

But $Ad_5=\dfrac{\sqrt{2}}{2}=\dfrac{1}{\sqrt{2}}$ and $\tan Ad_5M=m\sqrt{2}$.

$$\sin 54^\circ\, 44' = \frac{AD_5}{AO_1} = \frac{\sqrt{2}}{\sqrt{3}} \text{ and } \cos 54^\circ\, 44' = \frac{O_1D_5}{AO_1} = \frac{1}{\sqrt{3}}$$

$$\text{Hence } Ao_1 = \frac{1}{\sqrt{2}} \quad \frac{m\sqrt{2}}{\sqrt{\frac{1}{3}} + m\sqrt{2}\sqrt{\frac{2}{3}}} = \frac{m\sqrt{3}}{1+2m}$$

$$= \frac{m}{1+2m} AO_1;$$

$$\text{Or } \frac{Ao_1}{AO_1} = \frac{m}{1+2m} = \frac{1}{1+1+\frac{1}{m}}$$

48. If we call the distances 1, 1, and m, at which each of the 24 faces of the three-faced octahedron if produced would cut three of the semi-cubical axes at right angles to each other, indices; then the ratio of $\frac{Ao_1}{AO_1}$ = unity divided by the sum of the reciprocals of the indices. Calling R this ratio, then when $m=2$ $R=\frac{2}{5}$; $m=3$ $R=\frac{3}{7}$; $m=\frac{3}{2}$ $R=\frac{3}{8}$; $m=4$ $R=\frac{4}{9}$; $m=\frac{7}{4}$ $R=\frac{7}{18}$; $m=\frac{5}{4}$ $R=\frac{5}{14}$; and $m=\frac{65}{64}$ $R=\frac{65}{194}$.

49. When $m=1$, the three-faced octahedron becomes the octahedron, and its three indices are 1, 1, and 1, and $R=\frac{1}{3}$.

Taking 1 1 m as the symbol for the three-faced octahedron, 1 1 1 must be taken as the symbol for the octahedron.

50. For the octahedron Naumann's symbol is O; Miller's, 1 1 1; Brooke, Levy, and Des Cloizeau's a^1.

51. When the third index becomes infinite, or, in other words, the face cuts two axes and is parallel to the third, then $m=\frac{1}{0}=\infty$, and $\frac{1}{m}=0$; and the three-faced octahedron is then the rhombic dodecahedron.

52. The three indices of the rhombic dodecahedron are, therefore, 1, 1, and ∞; and 1 1 ∞ becomes its symbol. Naumann's symbol is ∞O; Miller's, 1 1 0; Brooke's, &c., b^1.

To inscribe the four-faced Cube in the Cube.

53. (Fig. 37, Plate IV.) Describe the square $AC_1D_1C_2$ equal one-fourth of the square $O_1O_4O_8O_5$ (fig. 27, Plate IV.), this being a face of the cube in which the four-faced cube is to be inscribed. Join AD_1 (fig. 37, Plate IV.). Produce D_1C_1 to O_1, and C_2A to D_5. Make C_1O_1 and $AD_5=AD_1$. Join O_1D_5.

Produce AC_1 to M, and make $AM=m$, m being any whole number or fraction greater than unity. The particular value of m will determine the particular species of the four-faced cube, there being a distinct species for every value which can be assigned to m.

Join C_2M cutting AD_1 in d_1. Join C_1d_1, C_1d_2.

In AD_5 take $Ad_5 = Ad_1$. Draw d_5o_1 parallel to AM and cutting AO_1 in o_1.

Join C_1o_1.

Then (fig. 37, Plate IV.) represents the same lines and letters seen in perspective in (fig. 9, Plate II.), or the square $AC_1D_1C_2$ represents one-fourth of the section of the circumscribing cube through the centres of opposite edges of the cube, and the parallelogram $C_1O_1D_5A$ one-fourth of that through two opposite edges and two diagonals of opposite faces.

Taking, therefore, eight points, O_1o_1, O_2o_2, O_2o_3, &c., O_8o_8, in the octahedral axes of the circumscribing cube (fig. 9, Plate II.), each equal to O_1o_1 (fig. 37, Plate IV.) in the solid, or marking them in the perspective by proportional compasses as described in § 36. Join together C_1o_1, C_1o_2, C_2o_1, C_2o_5, &c.; and also o_1o_4, o_1o_5, &c., as in fig. 9, and we have the four-faced cube inscribed in the cube. Since in fig. 9, $o_1d_1 = o_4d_4$, and D_1d_1 represents D_1d_1, fig. 37, it is evident that every edge of the four-faced cube such as o_1o_4 is bisected by a rhombic axis D_1d_1 in the point d_1.

54. If (fig. 34, Plate IV.) we draw $o_4d_1 = o_1d_5$ (fig. 37), produce o_4d_1 to o_1, and make $d_1o_1 = d_1o_4$; on o_4o_1 as base describe an isosceles triangle $C_1o_4o_1$, having its equal sides C_1o_4, C_1o_1 each $= C_1o_1$ (fig. 37).

Then $C_1o_4o_1$ will represent on a plane surface a face of the four-faced cube; and a net of 24 of these faces all equal to each other when folded up will form a solid four-faced cube, which can be accurately inscribed in a skeleton cube whose edges are all equal to O_1O_4 (fig. 9, Plate II.).

55. If we compare fig. 37, Plate IV., with fig. 9, Plate II., we see that o_1d_5 is parallel to AC_1, and C_2d_1 cuts AC_1 produced in M, AM being taken equal to m. Hence, by similarity and symmetry of construction, we see that every face of the four-faced cube cuts one of the three cubical axes at a distance $= AC$, another at m times AC, and is parallel to the third. Hence, taking $AC = 1$, then $1\ m\ \infty$ may be taken as the symbol for the four-faced cube.

Unity, m, and ∞ being the three indices of this form.

56. If m be represented as a fraction by $\frac{h}{k}$, then $\infty\ O\ m$ is Naumann's symbol, $h\ k\ o$ Miller's, $b^{\frac{h}{k}}$ Brooke, Levy, and Des Cloizeau's.

57. $m = \frac{6}{5}$ occurs in crystals of pyrite; $m = \frac{5}{4}$ in perowskite; $m = \frac{4}{3}$ in diamond and perowskite; $m = \frac{3}{2}$ in argentite, blende,

diamond, pyrite, and perowskite; $m=2$ in argentite, copper, cobaltine, cuprite, fluor, gold, gersdorfitte, garnet, magnetite, pyrite, percylite, salt, and silver; $m=\frac{7}{3}$ in cubane; $m=\frac{5}{2}$ in copper and fluor; $m=3$ in amalgam, fahlerz, fluor, hauerite, and pyrite; $m=4$ in cobaltine and silver; $m=5$ in cuprite; $m=40$ in fluor.

58. When $m=1$, the symbol for the four-faced cube becomes $1\ 1\ \infty$, or the four-faced cube becomes the rhombic dodecahedron. When $m=\infty$, the symbol becomes $1\ \infty\ \infty$, which is that of the cube, each of whose faces cuts one of three cubical axes and is parallel to that of the other two.

59. Hence fig. 9, Plate II., shows that the four-faced cube is a form of an infinite number of species, the points such as o_1, o_2, &c., in the octahedral axes lying between $\frac{1}{2}AO_1$ when it is the rhombic dodecahedron, and O_1 when it becomes the cube.

Constructing fig. 14, the skeleton cube, in wires, and the octahedron as shown with the lines passing through o and d in elastic strings, as before; then by pulling symmetrically all the points o_1, o_2, &c., from $Ao_1=\frac{1}{2}AO_1$ up to O_1, all the forms of the four-faced cube, though infinite in number, will be represented to the eye in a finite space of time.

To obtain the Ratios of the Octahedral and Rhombic Axes of the four-faced Cube to those of the circumscribing Cube.

60. (Fig. 37, Plate IV.) $\tan MC_2A=\dfrac{AM}{AC_2}=\dfrac{m}{1}$

angle $D_1AC_2=45°$ by construction.

Hence in triangle Ad_1C_2, $d_1C_2A+C_2d_1A+45°=180°$.

$$C_2d_1A=135°-d_1C_2A.$$

Therefore

$$\sin C_2d_1A=\sin(135°-d_1C_2A).$$
$$=\cos(90°-135°+d_1C_2A)=\cos(d_1C_2A-45°);$$

But in triangle Ad_1C_2, $\dfrac{Ad_1}{AC_2}=\dfrac{\sin d_1C_2A}{\sin C_2d_1A}=\dfrac{\sin d_1C_2A}{\cos(d_1C_2A-45°)}$

$$=\frac{\sin d_1C_2A}{\cos d_1C_2A\cos 45°+\sin d_1C_2A\sin 45°}$$
$$=\frac{\tan d_1C_2A}{\sqrt{\tfrac{1}{2}}(1+\tan d_1C_2A)}=\frac{m\sqrt{2}}{1+m}$$

But $AC_2=1$ and $AD_1=\sqrt{2}$.

Therefore $Ad_1=\dfrac{m}{1+m}AD_1$.

But $Ad_5=Ad_1$ and $AD_5=AD_1$ and o_1d_5 is parallel to O_1D_5.

Therefore $\dfrac{Ao_1}{AO_1}=\dfrac{Ad_5}{AD_5}=\dfrac{m}{1+m}$ or $Ao_1=\dfrac{m}{1+m}AO_1$.

Hence we see that the ratios of the octahedral and rhombic axes of the inscribed four-faced cube to those of the circumscribing cube are each equal to $\frac{m}{1+m}$. Calling this ratio R, and putting it under the form $R=\frac{1}{1+\frac{1}{m}}$; we see that for the cube $m=\infty$, $R=1$; and for the rhombic dodecahedron $m=1$, and therefore $R=\frac{1}{2}$.

Hence for the four-faced cube R varies from 1 to $\frac{1}{2}$.

When $m=\frac{6}{5}$, $R=\frac{6}{11}$; $m=\frac{5}{4}$, $R=\frac{5}{9}$; $m=\frac{4}{3}$, $R=\frac{4}{7}$;
$m=\frac{3}{2}$, $R=\frac{3}{5}$; $m=2$, $R=\frac{2}{3}$; $m=\frac{7}{3}$, $R=\frac{7}{10}$;
$m=\frac{5}{2}$, $R=\frac{5}{7}$; $m=3$, $R=\frac{3}{4}$; $m=4$, $R=\frac{4}{5}$;
$m=5$, $R=\frac{5}{6}$; $m=40$, $R=\frac{40}{41}$.

61. *To inscribe the twenty-four faced trapezohedron in the cube.*

(Fig. 31, Plate IV.) Describe the square $AC_1D_1C_2$=one-fourth the face of the cube $O_1O_5O_8O_4$ (fig. 27). Join AD_1. Produce D_1C_1 to O_1, and C_2A to D_5. Make C_1O_1 and AD_5 each$=AD_1$. Join O_1D_5. Produce AC_1 to M_1, and take $AM=m$, AC_1 being 1, and m any whole number or fraction greater than unity. m determines the particular species of the twenty-four-faced trapezohedron.

Join C_2M meeting AD_1 in d_1. In AD_5 take $Ad_5=Ad_1$.

Join d_5M cutting AO_1 in o_1. Join C_1o_1 and C_1d_1.

Then in (fig. 11, Plate II.), describe fig. 27, Plate IV., and take the eight points, o_1, o_2, &c., o_8, in the octahedral axes so that $\frac{Ao_1}{AO_1}=\frac{Ao_2}{AO_2}$ (fig. 11), $=\frac{Ao_1}{AO_1}$ (fig. 31, Plate IV.). And the twelve points d_1, d_2, &c., d_{12}, in fig. 11, Plate II., so that $\frac{Ad_1}{AD_1}=\frac{Ad_2}{AD_2}=\frac{Ad_1}{AD_1}$ (fig. 31, Plate IV.), as described in § 36.

Then joining the points C, d, and o, as shown in (fig. 11, Plate II.) the twenty-four-faced trapezohedron will be inscribed in the cube.

62. If (fig. 39, Plate IV.) we describe a triangle having one of its sides $C_1o_1=C_1o_1$ (fig. 31), another side $C_1d_1=C_1d_1$ (fig. 31), and its third side $o_1d_1=o_1d_5$ (fig. 31);

Then, on the other side of the base C_1o_1 (fig. 39), describe the triangle $C_1d_2o_1$ similar and equal to the triangle $C_1d_1o_1$.

$C_1d_1o_1d_2$ will represent on a plane surface a face of the twenty-four-faced trapezohedron, and 24 of these faces, formed into a net and folded together will make a solid twenty-four-faced trapezohedron, which can be inscribed with a skeleton cube whose face $=O_1O_5O_8O_4$, fig. 27, in the position shown in (fig. 11, Plate II.).

63. Since (fig. 31) C_2d_1 cuts AM in M, and d_5o_1 cuts AC_1 also in M, and comparing this with fig. 11, Plate II., we see that every face of the twenty-four-faced trapezohedron cuts one cubical axis at a distance equal AC_1, and two other cubical axes at m times this distance.

Taking AC_1 as unity, we see that the three indices of the twenty-four-faced trapezohedron are 1, m, and m. Its symbol, therefore, is 1, m, m.

Representing m as a fraction by $\frac{h}{k}$, Naumann's symbol is $m\,O\,m$; Miller's h, k, k; Brooke, Levy, and Des Cloizeau's $a^{\frac{h}{k}}$.

64. $m=\frac{4}{3}$ occurs in crystals of galena and garnet; $m=\frac{3}{2}$ in argentite, gold, and tennantite; $m=2$ in amalgam, argentite, analcime, boracite, cuprite, dufrenoysite, eulytine, fahlerz, franklinite, fluor, gold, galena, garnet, leucite, pyrite, pyrochlore, sal-ammoniac, sodalite, smaltine, and tennantite; $m=\frac{9}{4}$ in perowskite; $m=\frac{8}{3}$ in fluor; $m=3$ in blende, copper, fahlerz, fluor, gold, galena, magnetite, pyrite, perowskite, pyrochlore, and spinelle; $m=4$ in sal-ammoniac and kerate; $m=5$ in galena; $m=6$ in magnetite; $m=10$ in magnetite; $m=12$ in blende; $m=16$ in galena and magnetite; $m=40$ in pharmacosiderite.

65. *To find the ratios of the rhombohedral and octahedral axes of the twenty-four-faced trapezohedron to those of the circumscribing cube.*

The right-hand side of the (fig. 31, Plate IV.) being the same by construction as that of (fig. 37, Plate IV.) for the four-faced cube.

$$Ad_1=\frac{1}{1+\frac{1}{m}}AD_1, \text{ or } \frac{Ad_1}{AD_1}=\frac{m}{m+1} \text{ as in § 60.}$$

But fig. 31, $Ad_5=Ad_1=\frac{m}{m+1}\sqrt{2}$.

$$\tan Ad_5M=\frac{AM}{Ad_5}=m\,\frac{m+1}{m\sqrt{2}}=\frac{m+1}{\sqrt{2}}$$

but $\sin O_1Ad_5=\frac{O_1D_5}{AO_1}=\frac{1}{\sqrt{3}}$ and $\cos O_1Ad_5=\frac{AD_5}{AO_1}=\frac{\sqrt{2}}{\sqrt{3}}$

$$\begin{aligned}\text{also } \sin Ao_1d_5 &= \sin\{180^\circ-(Ad_5M+O_1Ad_5)\}.\\ &= \sin(Ad_5M+O_1Ad_5).\\ &= \sin Ad_5M\cos O_1Ad_5+\cos Ad_5M\sin O_1Ad_5.\end{aligned}$$

But in triangle Ao_1d_5 $\quad \dfrac{Ao_1}{Ad_5}=\dfrac{\sin\, Ad_5M}{\sin\, Ao_1d_5}$

Therefore $Ao_1=\dfrac{\sin\, Ad_5M}{\sin\, Ao_1d_5}\, Ad_5$.

$$=\frac{\sin\, Ad_5M\,.\, m\sqrt{2}}{(m+1)(\sin Ad_5M \cos O_1Ad_5+\cos Ad_5M \sin O_1Ad_5)}$$

$$=\frac{m\sqrt{2}}{(m+1)\,(\cos\, O_1Ad_5+\cot\, Ad_5M\, \sin\, O_1Ad_5)}$$

$$=\frac{m\sqrt{2}}{(m+1)\left\{\frac{\sqrt{2}}{\sqrt{3}}+\frac{\sqrt{2}}{m+1}\cdot\frac{1}{\sqrt{3}}\right\}}=\frac{m\sqrt{3}}{m+1+1}$$

$$=\frac{1}{1+\frac{1}{m}+\frac{1}{m}}\, AO_1.$$

Hence the ratio of the rhombic axes of the twenty-four-faced trapezohedron to those of circumscribing cube, or $\dfrac{Ad_1}{AD_1}=\dfrac{1}{1+\frac{1}{m}}$; and the ratio of the octahedral axes of the twenty-four faced trapezohedron, or $\dfrac{Ao_1}{AO_1}=\dfrac{1}{1+\frac{1}{m}+\frac{1}{m}}$

66. Representing $\dfrac{Ad_1}{AD_1}$ as R_1, and $\dfrac{Ao_1}{AO_1}$ as R_2.

R_1=unity divided by the sum of the reciprocals of the first two indices taken in order of magnitude, and

R_2=unity divided by the sum of the reciprocals of the three indices.

$$\begin{array}{llll}
\text{When } m=\frac{4}{3} & R_1=\frac{4}{7} & \text{and} & R_2=\frac{2}{5}\\
m=\frac{3}{2} & R_1=\frac{3}{5} & & R_2=\frac{3}{7}\\
m=2 & R_1=\frac{2}{3} & & R_2=\frac{1}{2}\\
m=\frac{9}{4} & R_1=\frac{9}{13} & & R_2=\frac{9}{17}\\
m=\frac{8}{3} & R_1=\frac{8}{11} & & R_2=\frac{4}{7}\\
m=3 & R_1=\frac{3}{4} & & R_2=\frac{3}{5}\\
m=4 & R_1=\frac{4}{5} & & R_2=\frac{2}{3}\\
m=5 & R_1=\frac{5}{6} & & R_2=\frac{5}{7}\\
m=10 & R_1=\frac{10}{11} & & R_2=\frac{5}{6}\\
m=12 & R_1=\frac{12}{13} & & R_2=\frac{6}{7}\\
m=16 & R_1=\frac{16}{17} & & R_2=\frac{8}{9}\\
m=40 & R_1=\frac{40}{41} & & R_2=\frac{20}{21}
\end{array}$$

67. When $m=1$, $R_1=\frac{1}{2}$, and $R_2=\frac{1}{3}$, and the twenty-four-faced trapezohedron becomes the octahedron.

When $m=\infty$ and $\frac{1}{m}=0$, $R_1=1$ and $R_2=1$, and the twenty-four-faced trapezohedron becomes the cube.

Hence, referring to (fig. 11, Plate II.) we see that the twenty-four-faced trapezohedron is a variable form of an infinite number of species, varying from the octahedron as one limit to the cube as the other.

If we represent this passage as in the instances of the three-faced octahedron, § 41, and four-faced cube, § 59, we must raise the eight points o_1, o_2, &c., o_8, from o_1 equal $\frac{1}{3}$ AO_1 in the octahedron (fig. 14) to O_1, fig. 8; at the same time raising the points d_1, d_2, &c., d_{12} along the lines AD_1, AD_2, &c., from d_1, d_2, &c. (fig. 14), equal one-half AD_1, to the point D_1, D_2, &c. (fig. 8); taking care that the point d shall have such a relation to o that two adjacent triangles on each side of Co are in the same plane.

68. *To inscribe the six-faced octahedron in the cube.*

(Fig. 35, Plate IV.) Describe the square $AC_1D_1C_2$ equal one-fourth of the square $O_1O_4O_8O_5$ (fig. 27). Join AD_1. Produce D_1C_1 to O_1, and C_2A to D_5, making C_1O_1 and AD_5 equal AD_1. Join O_1D_5. Produce AC_1 to M and N. Taking $AC_1=1$, make $AM=M$ and $AN=n$; m being any whole number or fraction greater than unity, and n any whole number or fraction greater than m.

Join C_2M, cutting AD_1 in d_1. Take $AD_5=Ad_1$. Join d_5N, cutting AO_1 in o_1. Join C_1o_1.

Then, in fig. 27, take 12 points, d_1, d_2, &c., d_{12}, in AD_1, AD_2, &c., AD_{12}, so that $\frac{Ad_1}{AD_1}$, $\frac{Ad_2}{AD_2}$, &c., $\frac{Ad_{12}}{AD_{12}}$ are each equal to $\frac{Ad_1}{AD_1}$, fig. 35, which can be easily done with proportional compasses.

Also, in fig. 27, take eight points, o_1, o_2, &c., o_8, in AO_1, AO_2, &c., AO_8, so that $\frac{Ao_1}{AO_1}$, $\frac{Ao_2}{AO_2}$, &c., $\frac{Ao_8}{AO_8}=\frac{Ao_1}{AO_8}$, fig. 35.

Join the points C, d, and o as in (fig. 10, Plate II.), and the six-faced octahedron inscribed in the cube will be shown in perspective. In a model showing the solid six-faced octahedron inscribed in a skeleton cube, each of the lines O_1o_1, O_2o_2, &c., O_8o_8, will be equal O_1o_1 fig. 35, and each of the lines D_1d_1, D_2d_2, &c., $D_{12}d_{12}$, will be equal D_1d_1, fig. 35.

69. Fig. 36, Plate IV. Draw a triangle, $C_1o_1d_2$, such that C_1o_1, fig. 36, $=C_1o$, fig. 35; C_1d_2, fig. 36, $=C_1d_1$, fig. 35; and o_1d_2, fig. 36, $=o_1d_5$, fig. 35.

Then $C_1o_1d_2$ (fig. 36) is a face on a plane surface of the six-faced octahedron which can be inscribed in a cube, each of whose faces are equal $O_1O_4O_8O_5$, fig. 27.

Forty-eight triangles, similar and equal to $C_1o_1d_2$, arranged as a *net* and cut out of cardboard, will fold up into a solid model of the six-faced octahedron.

70. Each face of the six-faced octahedron, if produced, cuts one axis of the cube at the distance $=1$, another at the distance $=m$, and the third at a distance n from the centre of the cube.

The three quantities, 1, m, and n are termed the three indices of the six-faced octahedron.

Its symbol, therefore, is $1, m, n$; Naumann's symbol is nOm.

If the three fractions $1, \frac{1}{m}, \frac{1}{n}$ be brought to a common denominator, and the three numerators divided, if they possess any common factor, by that factor, be represented by h, k, l, these being whole numbers, then h, k, l is Miller's symbol, and $b^{\frac{1}{h}} b^{\frac{1}{k}} b^{\frac{1}{l}}$ is that of Brooke, Levy, and Des Cloizeau.

71. The form $1, \frac{64}{63}, 64$ occurs in garnet; $1, \frac{5}{4}, \frac{5}{3}$ in pyrite and gold; $1, \frac{4}{3}, 2$ in linneite; $1, \frac{4}{3}, 4$ in garnet; $1, \frac{15}{11}, \frac{15}{7}$ in linneite; $1, \frac{3}{2}, 3$ in amalgam, cobaltine, cuprite, diamond, fahlerz, garnet, hauerite, magnetite, and pyrite; $1, \frac{8}{5}, 8$ in pyrite; $1, \frac{5}{3}, 5$ in boracite and pyrite; $1, \frac{5}{3}, 10$ in pyrite; $1, 2, 4$ in fluor, gold, and pyrite; $1, 2, 10$ in pyrite; $1, \frac{11}{5}, \frac{11}{3}$ in fluor; $1, \frac{16}{7}, 4$ in fluor; $1, \frac{7}{3}, 7$ in fluor; $1, 3, \frac{21}{5}$ in magnetite; $1, 4, 8$ in galena.

72. *To find the ratios of the rhombohedral and octahedral axes of the six-faced octahedron to those of the circumscribing cube.*

In fig. 35, Plate IV., the sides of the square $AC_1D_1C_2$ are by construction equal to unity. Hence $AD_1=\sqrt{2}$, and angle $D_1AC_2=45^\circ\ 0'$. Also $AM=m$ by construction. Let angle $AC_2d_1=a$. Then $Ad_1C_2=180^\circ-(a+45)$.

Then $\cos a=\dfrac{AC_2}{m}=\dfrac{1}{m}$,

and in triangle Ad_1C_2,

$$\frac{Ad_1}{AC_2}=\frac{\sin a}{\sin\{180-(a+45)\}}=\frac{\sin a}{\sin(a+45)}$$

$$Ad_1=\frac{\sin a}{\sin a \cos 45+\cos a \sin 45}=\frac{1}{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{2}}\cos a}$$

$$=\frac{\sqrt{2}}{1+\frac{1}{m}}=\frac{1}{1+\frac{1}{m}}\ AD_1.$$

Therefore $\frac{Ad_1}{AD_1}=\frac{1}{1+\frac{1}{m}}$.

Hence the ratio of each rhombic axis of the six-faced octahedron to that of the circumscribing cube is $\frac{1}{1+\frac{1}{m}}$, or of unity divided by the sum of the reciprocals of the two smaller indices of the six-faced octahedron.

73. Again in (fig. 35, Plate IV.), in the parallelogram $C_1O_1D_5A_1$, $C_1A=O_1D_5=1$, and $C_1O_1=AD_5=\sqrt{2}$; also $Ad_5=Ad_1$ $=\frac{\sqrt{2}}{1+\frac{1}{m}}$

Let $\gamma=O_1AD_5$ and $\beta=Ad_5N$. Then $Ao_1\dot{d}_5=180°-(\beta+\gamma)$.

But $AO_1^2=O_1D_5^2+AD_5^2=1+2=3$.

and $AO_1=\sqrt{3}$.

Also $\sin\ \gamma=\frac{O_1D_5}{AO_1}=\frac{1}{\sqrt{3}}$ and $\cos\ \gamma=\frac{AD_5}{AO_1}=\frac{\sqrt{2}}{\sqrt{3}}$

In triangle NAd_5 $\cot\beta=\frac{Ad_5}{AN}=\frac{Ad_5}{n}$

Also in triangle Ao_1d_5.

$$\frac{Ao_1}{Ad_5}=\frac{\sin\beta}{\sin\{180-(\beta+\gamma)\}}=\frac{\sin\beta}{\sin(\beta+\gamma)}$$

$$=\frac{\sin\beta}{\sin\beta\cos\gamma+\cos\beta\sin\gamma}=\frac{1}{\cos\gamma+\cot\beta\sin\gamma}$$

Hence $Ao_1=\frac{Ad_5}{\frac{\sqrt{2}}{\sqrt{3}}+\frac{Ad_5}{n}\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}}{\frac{\sqrt{2}}{Ad_5}+\frac{1}{n}}$

$$=\frac{\sqrt{3}}{\left(\frac{1+\frac{1}{m}}{\sqrt{2}}\right)\sqrt{2}+\frac{1}{n}}=\frac{AO_1}{1+\frac{1}{m}+\frac{1}{n}}$$

And $\frac{Ao_1}{AO_1}=\frac{1}{1+\frac{1}{m}+\frac{1}{n}}$

Hence ratio of the octahedral axis of six-faced octahedron is to that of the circumscribing cube as $\frac{1}{1+\frac{1}{m}+\frac{1}{n}}$, or unity divided by the sum of the reciprocals of its three parameters.

74. Let $R_1=\frac{Ao}{AO}$, and $R_2=\frac{Ad}{AD}$

For the form	1	$\frac{64}{63}$	64	$R_1=\frac{64}{127}$	$R_2=\frac{1}{2}$
	1	$\frac{5}{4}$	$\frac{5}{3}$	$R_1=\frac{5}{9}$	$R_2=\frac{5}{12}$
	1	$\frac{4}{3}$	2	$R_1=\frac{4}{7}$	$R_2=\frac{4}{9}$
	1	$\frac{4}{3}$	4	$R_1=\frac{4}{7}$	$R_2=\frac{1}{2}$
	1	$\frac{15}{11}$	$\frac{15}{7}$	$R_1=\frac{15}{26}$	$R_2=\frac{15}{33}$
	1	$\frac{3}{2}$	3	$R_1=\frac{3}{5}$	$R_2=\frac{1}{2}$
	1	$\frac{8}{5}$	8	$R_1=\frac{8}{13}$	$R_2=\frac{4}{7}$
	1	$\frac{5}{3}$	5	$R_1=\frac{5}{8}$	$R_2=\frac{5}{9}$
	1	$\frac{5}{3}$	10	$R_1=\frac{5}{8}$	$R_2=\frac{10}{17}$
	1	2	4	$R_1=\frac{2}{3}$	$R_2=\frac{4}{7}$
	1	2	10	$R_1=\frac{2}{3}$	$R_2=\frac{5}{8}$
	1	$\frac{11}{5}$	$\frac{11}{3}$	$R_1=\frac{11}{16}$	$R_2=\frac{11}{19}$
	1	$\frac{16}{7}$	4	$R_1=\frac{16}{23}$	$R_2=\frac{16}{27}$
	1	$\frac{7}{3}$	7	$R_1=\frac{7}{10}$	$R_2=\frac{7}{11}$
	1	3	$\frac{21}{5}$	$R_1=\frac{3}{4}$	$R_2=\frac{7}{11}$
	1	4	8	$R_1=\frac{4}{5}$	$R_2=\frac{8}{11}$

75. Referring now to (Plate II., fig. 10), we may observe that the six-faced octahedron is the form from which all the others represented on that plate are derived.

76. When the indices m and n are equal, and both greater than unity, the six-faced octahedron (fig. 10) becomes the twenty-four-faced trapezohedron, fig. 11, in which case two adjacent faces over the edge *Co* become in the same plane, and the 48 faces of the six-faced octahedron are reduced to the 24 faces of the twenty-four-faced trapezohedron.

77. When the index n becomes infinite, and m is some number or fraction greater than unity, the six-faced octahedron becomes the four-faced cube (fig. 9), and two adjacent planes over the edge *Cd* become in the same plane, and so the 48 faces of the six-faced octahedron are reduced to the 24 faces of the four-faced cube.

78. When the index m becomes unity, and n is some number or fraction greater than unity, the six-faced octahedron becomes the three-faced octahedron (fig. 13), and two adjacent faces over the edge *od* become in the same plane, and so the 48 faces of the six-faced octahedron are reduced to the 24 faces of the three-faced cube.

79. When the two indices m and n are both equal to unity, the six-faced octahedron becomes the octahedron (fig. 14), and the six faces round each octahedral axis become in the same plane, and the 48 faces of the six-faced octahedron are reduced to the eight faces of the octahedron.

80. When the index $m =$ unity, and n becomes infinite, the six-faced octahedron becomes the rhombic dodecahedron (fig. 12), and the four faces surrounding the rhombic axes are

in the same plane, and the 48 faces of the six-faced octahedron are reduced to the twelve faces of the rhombic dodecahedron.

81. When both the indices m and n become infinite, the six-faced octahedron becomes the cube fig. 8, and the eight faces surrounding the cubical axes are in the same plane, and the 48 faces of the six-faced octahedron are reduced to six faces of the cube.

82. By giving the necessary values to m and n, the formulæ belonging to any of the forms in Plate II. may be derived from those calculated for the six-faced octahedron. If fig. 10 be constructed, the outlines of the circumscribing cube in wire, and the 48 triangles Cdo in elastic strings fastened to the skeleton cube at C, and strings tying together the lines CdC and odo at d, and the four strings Cd meeting in o, and these be made to pass over pulleys at D and O; then by a proper adjustment of the lengths of Oo and Dd, taking care that the eight lines Oo and the twelve lines Dd are the same in length for each particular form,—the 48 triangles of the elastic six-faced octahedron may be made to assume the shape of any holohedral form of the cubical system.

83. Whenever faces parallel to different forms of crystals occur in the same crystal, such as is shown in a crystal of native copper (fig. 29*, Plate IV.*), these faces are always parallel to those of their respective forms when inscribed in a cube, every other form having the same invariable position with respect to the cube, as shown in (Plate II.) Faces parallel to those of the cube are marked C_1, C_2, C_3; octahedron o_1, o_4, o_8, o_5; rhombic dodecahedron d_1, d_2, d_5, &c., and H_1, H_2, &c., those of a four-faced cube are all shown on the same crystal.

84. It will also be seen by reference to (fig. 29), that the intersections of the faces of the crystal or the edges between C_1, H_6, d_1, H_5, C_2, H_8, d_9, and H_9 are lines parallel to one another, as also are those of C_3, H_1, d_5, H_5, C_2, H_7, d_8, H_{12}. Faces whose intersections are thus parallel are said to belong to the same zone, for a reason to be shown presently.

85. (Fig. 30*, Plate IV.*) Let the three planes $CDGH$, $DEKH$, and $EFLK$ be perpendicular to the plane $GHKL$, intersecting it in the lines GH, HK, and KL. From A, a point in the plane $GHKL$, draw AM perpendicular to GH, AN to HK, and AO to KL. Through A draw AB perpendicular to the plane $GHKL$. Then it may be easily shown by the Eleventh Book of Euclid, that CG, DH, EK, and FL are parallel to AB; also that AM is perpendicular to the plane $CDHG$, AN to $DEKH$, and AO to $EKLF$. Also DH perpendicular to GH and HK, and EK perpendicular to KH and KL.

AM, AN, and AO are called normals from the point O to the plane to which they are respectively perpendicular.

Now the inclination of the plane $CDHG$ to the plane $DHEK$ over their intersecting edge DH is measured by the angle MHN, MH and HN being drawn through the point H, perpendicular in each of the planes to their common intersection DH. Similarly the angle NKO measures the inclination of the plane $DEKH$ to the plane $EKLF$ over the edge of their intersection EK.

In every quadrilateral lineal figure drawn in the same plane the four angles of the figure are always equal to four right angles, and in the plane $GHKL$ the angles AMH, ANH, ANK, and AOK are all right angles. Hence the angle $MHN = 180° - MAN$, and the angle $NKO = 180° - NAO$.

In other words, the *normals* drawn through a point perpendicular to two intersecting planes, make with each other an angle which is the supplement to that which measures the inclination of these planes to each other over their intersecting edge.

86. The power of representing the combination of faces of crystals with each other such as (fig. 29*, Plate IV.*) is necessarily limited to those of comparatively few faces. But, taking advantage of the relationship of the inclination of faces of crystals measured over their edges of intersection to that of their normals drawn from a certain point within the crystal, Professor Neumann, of Königsberg, devised a system by which the relationship of all the forms of any number of crystals might be graphically represented at one view.

For instance, to represent the relationship of all the forms of the cubical system to each other, we suppose the cube (fig. 27, Plate IV.) to be inscribed in a sphere whose centre corresponds with A, the centre of the cube. From this centre A, normals are drawn perpendicular to every face of the cube, and to those of every form which can be inscribed in it.

The points where these normals cut the surface of the circumscribing sphere are called the poles of their respective faces, and the arc of the great circle between any two poles is the supplement of that arc which measures the inclination of their respective faces over the straight edge of their intersection.

87. Referring to (fig. 27, Plate IV.), we see that AC_1 and AC_2, the normals of opposite faces of the cube, are in the same straight line, as also are AC_2 and AC_4, AC_3 and AC_5; also that the three axes C_1C_6, C_2C_4, and C_3C_5 are perpendicular to each other. The six equal lines AC_1, AC_2, &c., AC_6 are equal radii of a sphere, which can be inscribed in the cube, having A for its centre and touching the six faces of the cube in their poles, C_1, C_2, &c., C_6.

Upon this sphere we may project the poles of all the faces of the different forms (fig. 9 to fig. 14, Plate II.), which can be inscribed in the cube.

Let (fig. 31* and fig. 32*, Plate IV.*) represent the projections of two hemispheres of this sphere upon the plane of the paper.

Let C_1C_6 and C_5C_3 (fig. 31*) be two diameters intersecting at right angles in C_2. Also C_1C_6 and C_5C_4 (fig. 32*) be two diameters intersecting at right angles in C_4.

Then C_1, C_2, C_3, &c., C_6, represent the poles of the six faces of the cube on the sphere of projection. Also the eight equilateral spherical triangles $C_1C_2C_3$, $C_1C_5C_2$, $C_5C_2C_6$, &c., divide the sphere of projection into eight equal octants.

88. Bisect each of the twelve arcs C_1C_2, C_1C_3, C_1C_4, C_1C_5, &c., by the points D_1, D_2, D_3, and D_{12}; these twelve points will be the twelve poles of the rhombic dodecahedron on the sphere of projection (figs. 31* and 32*, Plate IV.*), or the twelve points where the rhombic axes AD_1, AD_2, AD_3, AD_4, &c., of fig. 27 cut the surface of the sphere of projection inscribed in the cube.

89. Join C_1D_5, C_2D_2, C_3D_1 by arcs of great circles meeting in O_1; this will divide the octant of the sphere $C_1C_2C_3$ into six equal and similar spherical triangles. Let this be done to each of the other octants. Then (fig. 31* and fig. 32*, Plate IV.*) the eight points O_1, O_2, &c., O_8, will represent the eight poles of the octahedron on the sphere of projection.

The sphere of projection is thus divided into 48 equal and similar but right and left-handed spherical triangles, indicated by the triangles COD, with different indices to the letters.

90. Any great circle of the sphere of projection is called a zone circle, and the poles of all faces which are in that great circle are said to lie in the same zone, and their intersections will be parallel to each other (see § 84 and 85).

91. We see in (fig. 9, Plate II.) that the normal to any face such as $C_1o_1o_2$, must, by the symmetry of construction of the four-faced cube, pass through some point in the line C_1d_2. Hence in the sphere of projection (figs. 31* and 32*, Plate IV.*), the 24 poles of any four-faced cube will lie in each of the 24 arcs CD.

92. The normals to any face of the twenty-four-faced trapezohedron, such as $C_1d_1o_1d_2$ (fig. 11, Plate II.), must, by symmetry of construction, pass through the line C_1o_1. Hence in the sphere of projection (figs. 31* and 32*, Plate IV.*), the

24 poles of any twenty-four-faced trapezohedron will lie in each of the 24 arcs CO.

93. The normals to any face of the three-faced octahedron (fig. 13, Plate II.), such as $C_1o_1C_2$, must, by symmetry of construction, pass through the line d_1o_1. Hence in the sphere of projection, (figs. 31 and 32, Plate IV.), the 24 poles of the three-faced octahedron will lie in each of the arcs DO.

94. Hence in the same zone $C_1D_1C_2D_9C_6D_{11}C_4D_3$ there will be four poles of the cube, C_1, C_2, C_6, C_4; four poles of the rhombic dodecahedron, D_1, D_9, D_{11}, D_3; and eight poles of the four-faced cube.

The same will be true of the two zones $C_2D_5C_3$ and $C_3D_2C_1$.

Again in the zone $C_3O_1D_1O_4C_5O_6D_{11}O_7C_5$, there will be two poles of the cube, C_3 and C_5, two poles of the rhombic dodecahedron, D_1 and D_{11}, four of the octahedron, O_1, O_4, O_6, and O_7, four of the three-faced octahedron, and also four of the twenty-four-faced trapezohedron, will lie.

The same will also be true for the five other zones, $C_3O_5D_9$, $C_1O_1D_5$, $C_1O_4D_8$, $C_2O_1D_2$, and $C_2O_4D_4$.

95. The 48 poles of any six-faced octahedron will, from the symmetry of its construction, occupy similar positions within the 48 spherical triangles CDO (figs. 31* and 32*, Plate IV.*).

96. In each of the 48 spherical triangles CDO (figs. 31 and 32, Plate IV.*) is marked a notation for each of the 48 poles of the six-faced octahedron in terms of its three indices. The order in which the three indices 1, m, and n are written, mark the distances at which the face of the six-faced octahedron corresponding to the pole marked on the sphere of projection, cuts each of three cubical axes taken in the order AC_3, AC_2, and AC_1 (fig. 27, Plate IV.). When the index has a negative sign placed over it, it signifies that it cuts the axis AC_3 produced in the direction AC_5, AC_2 in AC_4, or AC_1 in AC_6.

Thus the spherical triangle $C_2D_5O_1$ (fig. 31*, Plate IV.*) has marked in it the indices m, 1, n, which indicates that the face $C_2d_5o_1$ of the six-faced octahedron (fig. 3, Plate I.) cuts the axis AC_3 produced at the distance $m \times AC_3$, the axis AC_2 at the point C_2, and the third axis AC_1 produced, at $n \times AC_1$.

Again the indices $\bar{n}\,1\,\bar{m}$, in the triangle $C_2O_8D_9$ (fig. 31*, Plate IV.*), show that the face $C_2o_8d_9$ of the six-faced octahedron (fig. 3, Plate I.) cuts the axis AC_5 produced at a distance $n \times AC_5$, the axis AC_2 at the point C_2, and the axis AC_6 at a distance $m \times AC_6$.

97. The indices marked on (figs. 31* and 32*, Plate IV.*), enable us readily to find the notation for any face of any form in Plate II.

In (fig. 31*, Plate IV.*) the indices $m\,1\,\bar{n}$ in triangle $C_2o_5d_5$

signify that the face of the six-faced octahedron marked $C_2o_5d_5$ (fig. 10, Plate II.) cuts the axis AC_3 at a distance m from A, the axis AC_2 at C_2, and AC_6 at a distance n from A.

The indices $m\ 1\ n$ in the triangle $C_2O_1D_5$ indicate that the face of the six-faced octahedron marked $C_2o_1d_5$, fig. 10, Plate II., cuts AC_3 at a distance m, AC_2 at C_2, and AC_3 at a distance n from A.

98. Hence n without any sign over it signifies that the face of the six-faced octahedron which it indicates cuts the cubic axis C_1AC_6 in the direction of AC_1 produced; if it has the sign — placed over it, it signifies that the face cuts the axis in the direction of AC_6 produced.

Now if m be infinite, represented by the symbol ∞, or $\frac{1}{o}$, this signifies that the face cuts the axis neither in the direction AC_1 nor AC_6, and that if produced ever so far in either direction it will not cut the axis C_1AC_6, and is therefore parallel to it. Hence when $m=\infty$, $\overline{m}$ and m indicate that the face is parallel to the axis, to AC_3 if m is in the first place, to AC_2 if in the second, and to AC_1 if in the third place.

99. Now, if in the triangle $C_2D_5O_5$ (fig. 31*, Plate IV.*), whose indices are $m\ 1\ \overline{n}$, we make both m and n infinite, since $\overline{\infty}$ and ∞ are the same, we see that $\infty\ 1\ \infty$ is the index of the face $O_1O_4O_3O_5$ of the cube (fig. 1, Plate I.); also that, substituting the sign ∞ for both m and $\overline{n}$, the same notation $\infty\ 1\ \infty$ stands for each of the eight triangles $C_2O_1D_5$, $C_2O_1D_1$, $C_2O_4D_1$, $C_2O_4D_8$, $C_2O_8D_8$, $C_2O_8D_9$, $C_2O_5D_9$, and $C_2O_5D_5$.

100. When n alone is infinite in the index $m\ 1\ \overline{n}$, $m\ 1\ \infty$ is the index of both $C_2o_5d_5$ and $C_2o_1d_5$, or of the face $C_2o_1o_5$ of the four-faced cube (fig. 9, Plate II.).

101. When $n=\infty$, and $m=1$, the index $m\ 1\ \overline{n}$ becomes $1\ 1\ \infty$, which is the symbol for the four triangles $C_2d_5o_5$, $C_3d_5o_5$, $C_2o_1d_5$, and $C_3o_1d_5$, or of the face $C_2o_1C_3o_5$ of the rhombic dodecahedron (fig. 12, Plate II.).

102. When $n=m$, the index $m\ 1\ \overline{n}$ becomes $m\ 1\ \overline{m}$, which is that of the two triangles $C_2o_5d_5$ and $C_2d_9o_5$, or of the face $C_2d_9o_5d_5$ of the twenty-four faced trapezohedron (fig. 11, Plate II.).

103. When $m=1$, the index $m\ 1\ \overline{n}$ becomes $1\ 1\ \overline{n}$, which is that of the two triangles $C_2o_5d_5$, $C_3o_5d_5$, or of the face $C_2o_5C_3$ of the three-faced octahedron (fig. 13, Plate II.).

104. When $m=1$ and $n=1$, the index $m\ 1\ \overline{n}$ becomes $1\ 1\ \overline{1}$, which is the same for the six triangles, $C_2o_5d_5$, $C_3o_5d_5$, $C_8o_5d_{10}$, $C_6o_5d_{10}$, $C_6o_5d_9$, and $C_2o_5d_9$, or of the face $C_2C_3C_6$ of the octahedron (fig. 14, Plate II.).

105. To find the normal to a plane from the centre of the cubical axes in terms of the indices of that plane.

Let BCD (fig. 33*, Plate IV.*) be a plane cutting the three cubical axes AB, AC, and AD, in the points B, C, and D. Let $AB=a$, $AC=b$, and $AD=c$, be the three indices of this plane.

Through A draw AE perpendicular to BC in triangle ABC. Join ED.

Through A draw AF perpendicular to DE in triangle ADE.

Then AF is perpendicular to the plane ABC. Let $AF=R$, then R is the normal drawn through A to the plane whose indices are a, b, c.

Through F in triangle ADE draw FG perpendicular to AE, and in triangle ABC draw GH perpendicular to AB.

Let $AH=x$, $GH=y$, and $FG=z$, are called the rectangular co-ordinates of the point F, referred to the rectangular axes AB, AC, AD, or AX, AY, AZ (fig. 33*, Plate IV.*), is drawn in perspective. (Fig. 35*) is the triangle ACB of (fig. 33*), drawn on the plane of the paper; (fig. 34) the triangle DAE of the same figure, also on the plane of the paper.

Let angle $AEF=\beta$. Then by construction $AFG=\beta$, $DAF=\beta$, $ADF=90°-\beta$, and $FAE=90°-\beta$.

$z=FG=AF \sin FAG=R \cos \beta$.

Also $R=AF=AD \sin ADF=c \cos \beta$.

Hence $z=\dfrac{R}{c}$.

Again, in triangle AGF, $AG=AF \sin AFG=R \sin \beta$.

Also in triangle ABC, let α=angle ABC, then by construction $CAE=\alpha$, $AGH=\alpha$, $ECA=90-\alpha$, and $EAB=90-\alpha$.

In triangle AGH, $x=AH=AG \sin AGH=AG \sin \alpha=$ $R \sin \beta \sin \alpha$.

Also in triangle AEB, $AE=a \sin \alpha$ and $\sin \alpha=\dfrac{AE}{a}$

In triangle AFE, $R=AF=AE \sin \beta$ and $\sin \beta=\dfrac{R}{AE}$

But $x=R \sin \beta \sin \alpha=R\,.\,\dfrac{R}{AE}\,.\,\dfrac{AE}{a}=\dfrac{R^2}{a}$

Again in triangle AGH, $y=GH=AG \cos \alpha=R \sin \beta \cos \alpha$.

In triangle ACE,

$$AE=AC \cos CAE=b \cos \alpha\text{; and } \cos \alpha=\frac{AE}{b}$$

But $\sin \beta=\dfrac{R}{AE}$

Hence $y=R \sin \beta \cos \alpha=R\,.\,\dfrac{R}{AE}\,.\,\dfrac{AE}{b}=\dfrac{R^2}{b}$

Hence $x = \frac{R^2}{a}$, $y = \frac{R^2}{b}$, and $z = \frac{R^2}{c}$

In triangle AGF, $R^2 = AF^2 = FG^2 + AG^2 = z^2 + AG^2$.

And in triangle AGH, $AG^2 = AH^2 + HG^2 = x^2 + y^2$.

Hence $R^2 = x^2 + y^2 + z^2 = \frac{R^4}{a^2} + \frac{R^4}{b^2} + \frac{R^4}{c^2}$

And $R^2 = \dfrac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$ $\quad R = \dfrac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$

106. In (fig. 33*, Plate IV.*), join CF and BF. Then because $AF = R$ is perpendicular to the plane BCD, AF is perpendicular to CF and BF as well as DF.

$$\text{Therefore } \cos FAD = \frac{R}{c} = \frac{\frac{1}{c}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\text{also } \cos FAB = \frac{R}{a} = \frac{\frac{1}{a}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\text{and } \cos FAC = \frac{R}{b} = \frac{\frac{1}{b}}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}.$$

Where FAD, FAB, and FAC are the three angles which the normal makes with the three cubical axes which it cuts at the distances a, b, and c.

107. Given the indices of any two faces of a crystal of the cubical system, find the angle between their two normals at the centre of cube, or the supplement of the angle of inclination of these two faces over the edge of their intersection.

(In fig. 36*, Plate IV.*)

Let $AF = R$ be the normal to the plane whose indices are a, b, c.

$AF_1 = R_1$ be the normal to the plane whose indices are a_1, b_1, c_1.

Let $x = AH$, $y = HG$, and $z = FG$ be the rectanglar co-ordinates of the point F (*see* § 105) referred to the rectangular axes AX, AY, AZ.

And $x_1 = AH_1$, $y_1 = H_1G_1$, $z_1 = F_1G_1$, similar co-ordinates for the point F_1.

Fig. 36* is drawn in perspective. Fig. 37* is the plane

F_1G_1GF of (fig. 36*) drawn on the plane of the paper. (Fig. 38*) the plane $YAHH_1G_1$ also drawn on the plane of the paper.

Join FF_1 and GG_1. In plane FF_1G_1G draw KF parallel to GG_1, and therefore perpendicular F_1G_1; also in plane GG_1H_1H draw GL parallel to HH_1. Then $KFGG_1$ and $HGLH_1$ are rectangular parallelograms and their opposite sides are equal.

Then (fig. 37*) $FF_1^2=F_1K^2+KF^2=(F_1G_1-KG_1)^2+G_1G^2.$
$$=(F_1G_1-FG)^2+G_1G^2=(z_1-z)^2+G_1G^2.$$

But (fig. 38*)
$$G_1G^2=GL^2+G_1L^2=HH_1^2+(G_1H_1-LH_1)^2.$$
$$=(AH_1-AH)^2+(G_1H_1-GH)^2=(x_1-x)^2+(y_1-y)^2.$$

And $FF_1^2=(x_1-x)^2+(y_1-y)^2+(z_1-z)^2.$

We have seen (§ 105) that $R^2=x^2+y^2+z^2$, and that
$$x=\frac{R^2}{a},\quad y=\frac{R^2}{b},\quad \text{and } z=\frac{R^2}{c}$$

Similarly $R_1^2=x_1^2+y_1^2+z_1^2$, and $x_1=\frac{R_1^2}{a}$, $y_1=\frac{R_1^2}{b}$, and $z_1=\frac{R_1^2}{c}$

In triangle FF^1A, fig. 39, if we put θ for the angle FAF_1 or the angle between the normals AF, AF_1, or R and R_1 at the point A; we have
$$FF_1^2=AF_1^2+AF^2-2AF_1\,.\,AF\cos\theta=R_1^2+R^2-2RR_1\cos\theta;$$
but $FF_1^2=(x_1-x)^2+(y_1-y)^2+(z_1-z)^2.$

Hence $(x_1-x)^2+(y_1-y)^2+(z_1-z)^2=R_1^2+R^2-2RR_1\cos\theta;$

or $x_1^2-2xx_1+x^2+y_1^2-2y_1y+y^2+z_1^2-2z_1z+z^2=R_1^2+R^2-2RR_1\cos\theta.$

But $R_1^2=x_1^2+y_1^2+z_1^2$ and $R^2=x^2+y^2+z^2.$

Hence $x_1x+y_1y+zz_1=RR_1\cos\theta,$

or $$\frac{R^2R_1^2}{aa_1}+\frac{R^2R_1^2}{bb_1}+\frac{R^2R^{12}}{cc_1}=RR_1\cos\theta.$$
$$\cos\theta=RR_1\left(\frac{1}{aa_1}+\frac{1}{bb_1}+\frac{1}{cc_1}\right),$$

but $R^2=x^2+y^2+z^2=\frac{R^4}{a^2}+\frac{R^4}{b^2}+\frac{R^4}{c^2}$ and $R^2=\dfrac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

also $R_1^2=\dfrac{1}{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}$

Therefore $$\cos\theta=\frac{\frac{1}{aa_1}+\frac{1}{bb_1}+\frac{1}{cc_1}}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}\right)}}$$

108. In fig. 33, Plate IV.*, let p_1 = angle FAB, which the normal AF makes with the axis AX; p_2 = angle FAC, which the normal makes with the axis AY; and p_3 = angle FAD makes with AZ.

AX is the normal to a face of the cube which cuts the axis AX at a, AY at ∞, and AZ at ∞; or $a_1 = a$, $b_1 = \infty = \frac{1}{o}$, and $c_1 = \infty = \frac{1}{o}$

$$\text{and } \cos p_1 = \frac{\frac{1}{a^2}}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\frac{1}{a^2}}} = \frac{\frac{1}{a}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

AY is the normal to a face of the cube, or a plane whose indices are $a_1 = \frac{1}{o}$, $b_1 = b$, and $c_1 = \frac{1}{o}$

$$\cos p_2 = \frac{\frac{1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

AZ is the normal to a plane whose indices are $a_1 = \frac{1}{o}$, $b_1 = \frac{1}{o}$, and $c_1 = c$,

$$\text{and } \cos p_3 = \frac{\frac{1}{c}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

The same formulæ we obtained in § 106.

109. If p_1, p_2, p_3 be the angles which the normal to the plane whose indices are $a\ b\ c$, makes with the three axes AX, AY, and AZ;

Also, q_1, q_2, q_3 the angles the normal to the plane whose indices are $a_1\ b_1\ c_1$, makes with the same axes,

$$\text{Then } \cos p_1 = \frac{\frac{1}{a}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \quad \cos p_2 = \frac{\frac{1}{b}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\text{and } \cos p_3 = \frac{\frac{1}{c}}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}};$$

$$\text{and } \cos q_1 = \frac{\frac{1}{a_1}}{\sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}} \quad \cos q_2 = \frac{\frac{1}{b_1}}{\sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}}$$

$$\text{and } \cos q_3 = \frac{\frac{1}{c_1}}{\sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}}$$

Substituting these values in the expression

$$\cos\theta = \frac{\frac{1}{aa_1}+\frac{1}{bb_1}+\frac{1}{cc_1}}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}\right)}}$$

we have

$$\cos\theta = \cos p_1 \cos q_1 + \cos p_2 \cos q_2 + \cos p_3 \cos q_3.$$

110. If, in (figs. 31* and 32*), we substitute for $1, m, n$; $\cos p_1$, $\cos p_2$, and $\cos p_3$ in the order in which they occur, we have a notation for every face of the six-faced octahedron in terms of p_1, p_2, and p_3, the polar distances of the face from the three adjacent poles of the cube; -1, $-m$, and $-n$ being replaced by $-\cos p_1$, $-\cos p_2$, and $-\cos p_3$.

Thus if θ be the angle between the normals of the faces whose poles lie in the spherical triangles $C_1D_1O_1$ and $C_2O_1D_2$, or the supplement of the angle of their inclination over the edge C_1o_1 (fig. 3, Plate I.),

$$\cos\theta = \frac{\frac{1}{mn}+\frac{1}{mn}+1}{\sqrt{\left(\frac{1}{n^2}+\frac{1}{m^2}+1\right)\left(\frac{1}{m^2}+\frac{1}{n^2}+1\right)}} = \frac{\frac{2}{mn}+1}{1+\frac{1}{m^2}+\frac{1}{n^2}}$$

if expressed by the indices of the six-faced octahedron.

$$\cos\theta = \cos p_3 \cos p_2 + \cos p_2 \cos p_3 + \cos p_1 \cos p_1$$
$$= 2 \cos p_3 \cos p_2 + \cos^2 p_1$$

if expressed by the three polar distances of the pole of any face from the three adjacent poles of the cube.

111. The notation for each face of a crystal, or of its pole on the sphere of projection, is expressed in the terms of the three indices at which a plane drawn through a point in one of the cubical axes, taken at an arbitrary distance called unity from the centre where the axes meet, cuts the other two axes which are at right angles to the former; the indices being reckoned positive or negative as the points of intersection are right or left of A along the three axes AC_1, AC_2, and AC_3.

112. The relations of any pole to any other pole, and other problems relating to crystals, can therefore be solved by that branch of Geometry of Three dimensions which relates to the properties of the plane and straight line. This method is used by Professor Naumann, of Freiberg, in his works on crystallography.

113. The use of the sphere of projection has led to that of spherical trigonometry for solving all questions of crystallography, retaining, however, the notation for the faces of crystals in terms of the indices of the plane cutting the axes derived from the geometry of the plane. Professor Miller, of Cambridge, uses Spherical Trigonometry in his works on crystallography.

114. The position of any pole on the sphere of projection may be determined by its polar distance from a definite pole on the sphere corresponding to the north pole of the terrestrial sphere, and its longitude by an arc measured along the equator of the fixed pole, from a definite point in that equator. Just as the position of any point on the earth's surface is determined by its latitude and longitude.

In the crystallographic sphere of projection it is more convenient to use the polar distance instead of the latitude; the polar distance being an arc 90° less than that of the latitude.

115. The forms of the cubical system possess the highest degree of symmetry, each face of every form being symmetrical right and left from the centre to each of the three cubical axes. Hence we have seen that the three indices taken positive or negative, or right and left of the centre, give the notation or express this degree of symmetry.

116. In (figs. 31* and 32*, Plate IV.*), we see that if in the sphere of projection we take C_1 as the north pole and C_6 as the south, and $C_3C_2C_5C_4$ as the equator, and measure longitude from C_3.

If p be the north polar distance of the face $1\,m\,n$ and λ be its longitude,

Then p will be the north polar distance of the eight faces or poles $1\,m\,n$, $m\,1\,n$, $\bar{m}\,1\,n$, $\bar{1}\,m\,n$, $\bar{1}\,\bar{m}\,n$, $\bar{m}\,\bar{1}\,n$, $m\,\bar{1}\,n$, and $1\,\bar{m}\,n$, whose longitudes are λ, $90-\lambda$, $90+\lambda$, $180-\lambda$, $180+\lambda$, $270-\lambda$, $270+\lambda$, and $360-\lambda$.

Also p will be the south polar distance of the eight faces $1\,m\,\bar{n}$, $m\,1\,\bar{n}$, $\bar{m}\,1\,\bar{n}$, $\bar{1}\,m\,\bar{n}$, $\bar{1}\,\bar{m}\,\bar{n}$, $\bar{m}\,\bar{1}\,\bar{n}$, $m\,\bar{1}\,\bar{n}$, and $1\,m\,\bar{n}$, whose longitudes are respectively the same as the former.

Again, if we take C_2 as the north pole, C_4 as the south, and $C_1C_3C_6C_5$ as the equator, and measure the longitude from C_1, we have eight faces, $m\,n\,1$, $1\,n\,m$, $1\,n\,\bar{m}$, $m\,n\,\bar{1}$, $\bar{m}\,\bar{n}\,1$, $\bar{1}\,n\,\bar{m}$, $1\,n\,m$, and $\bar{m}\,n\,1$, having the same north polar distances and the same longitudes as the former.

Also eight more faces $m\bar{n}1$, $1\bar{n}m$, $1\bar{n}\bar{m}$, $m\bar{n}\bar{1}$, $\bar{m}\bar{n}\bar{1}$, $\bar{1}\bar{n}\bar{m}$, $\bar{1}\bar{n}m$, $\bar{m}\bar{n}1$, having the same south polar distance and longitudes as the former.

The 16 other faces will have the similar polar distances and longitudes, taking C_3 as the north and C_5 as the south pole, and $C_1C_2C_6C_4$ as the equator.

117. (Fig. 39*, Plate IV.*).—In the three rectangular cubical axes, take $AB=1$, $AM=m$, $AN=n$.

Through A draw AG perpendicular MB, AH perpendicular NC_3, and AK perpendicular MN.

Join NG, HM, and BK meeting in F. Join AF.

Since the normal from A or the perpendicular to the plane NMB must, by construction, lie in each of the three planes NAG, HAM, and KAB, AF, their common intersection, must be the normal to the plane NMB.

Hence AF is the normal to the plane whose notation is $1\,m\,n$. AG is the normal to a plane passing through MC_3 parallel to AN, or the normal to a face of the four-faced cube whose notation is $1\,m\,\infty$, AH the normal to $1\,\infty\,n$, AK to $\infty\,m\,n$ or $\infty\,1\,\frac{n}{m}$.

(Fig. 40*, Plate IV.*).—Let C_1, C_2, C_3 be the poles of the three rectangular or cubical axes, or the points where AN, AM, and AB of fig. 39* cut the sphere of projection.

Let h, k, and g be the points where AH, AK, and AG cut the sphere of projection. Join C_1g, C_3k, and C_2h by arcs of great circles meeting in f.

Then g is the pole of $1\,m\,\infty$, h of $1\,\infty\,n$, k of $\infty\,1\,\frac{n}{m}$, and f of $1\,m\,n$.

Let $fC_3=p_1$, $fC_2=p_2$, $fC_1=p_3$, $C_2k=\lambda_1$, $C_3h=\lambda_2$, $C_3g=\lambda_3$.

Then p_1, p_2, and p_3 will be the polar distances of the pole of $1\,m\,n$ from C_3, C_2, and C_1, taken in order of magnitude.

Comparing § 96 with (fig. 31*, Plate IV.*), the face $1\,m\,n$ cuts the axis AC_3 in B, AC_2 in M, and AC_1 in N to form (fig. 39*). Hence arc C_1f (fig. 40*) $=p_3$, and $C_3g=\lambda_3$, is its polar distance and longitude.

The face $1\,n\,m$ cuts the axis AC_3 in B, AC_2 in N, and AC_1 in M; and (fig. 40*) $C_2f=p_2$ and $C_3h=\lambda_2$, is its polar distance and longitude.

Also the face $m\,n\,1$ cuts the axis AC_3 in M, AC_2 in N, and AC_1 in B; and (fig. 40*) $C_3f=p_1$ and $C_2k=\lambda_1$, is its polar distance and longitude.

Calling (figs. 31* and 32*, Plate IV.*), C_1 the North pole, $C_3C_2C_5$ the equator, and measuring longitude from C_3, λ_3 will be the longitude of $1\,m\,n$, $90°-\lambda_3$ of $m\,1\,n$, $90°+\lambda_3$ of $\bar{m}\,1\,n$,

$180°-\lambda_3$ of $\bar{1}\,m\,n$, $180°+\lambda_3$ of $\bar{1}\,\bar{m}\,n$, $270°-\lambda_3$ of $\bar{m}\,\bar{1}\,n$, $270°+\lambda_3$ of $m\,\bar{1}\,n$, and $360°-\lambda_3$ of $1\,\bar{m}\,n$.

The north polar distances of these eight faces will each be p_3.

λ_3 the longitude of $1\,m\,\bar{n}$, $90°-\lambda_3$ of $m\,1\,\bar{n}$, $90°+\lambda_3$ of $\bar{m}\,1\,n$, $180°-\lambda_3$ of $\bar{1}\,m\,\bar{n}$, $180°+\lambda_3$ of $\bar{1}\,\bar{m}\,\bar{n}$, $270°-\lambda_3$ of $m\,\bar{1}\,\bar{n}$, $270°+\lambda_3$ of $m\,\bar{1}\,\bar{n}$, and $360°-\lambda_3$ of $1\,\bar{m}\,\bar{n}$.

The north polar distances of these eight faces will each be $180°-p_3$.

λ_2 will be the longitude of $1\,n\,m$, $90°-\lambda_2$ of $n\,1\,m$, $90°+\lambda_2$ of $\bar{n}\,1\,m$, $180°-\lambda_2$ of $\bar{1}\,n\,m$, $180°+\lambda_2$ of $\bar{1}\,\bar{n}\,m$, $270°-\lambda_2$ of $\bar{n}\,\bar{1}\,m$, $270°+\lambda_2$ of $n\,\bar{1}\,m$, $360°-\lambda_2$ of $1\,\bar{n}\,m$.

The north polar distances of these eight faces will each be p_2.

The eight similar faces in the southern hemisphere will have the same longitudes as those corresponding to them in the northern, the eight north polar distances being each equal $180°-p_2$.

λ_1 will be the longitude of $m\,n\,1$, $90°-\lambda_1$ of $n\,m\,1$, $90°+\lambda_1$ of $\bar{n}\,m\,1$, $180°-\lambda_1$ of $\bar{m}\,n\,1$, $180°+\lambda_1$ of $\bar{m}\,\bar{n}\,1$, $270°-\lambda_1$ of $\bar{n}\,\bar{m}\,1$, $270°+\lambda_1$ of $n\,\bar{m}\,1$, and $360°-\lambda_1$ of $m\,\bar{n}\,1$.

p_1 will be the north polar distance of each of these eight faces.

The corresponding eight faces of the southern hemisphere will have the same longitudes as the corresponding ones in the northern, $180°-p_1$ being the north polar distance of these eight faces.

Hence the 48 faces or poles of the six-faced octahedron can be expressed in terms of p_1, λ_1, p_2, λ_2, and p_3, λ_3; and, as all other forms of the cubical system can be derived from those of the six-faced octahedron, all faces of those forms can be similarly expressed.

118. Given p_3 and λ_3 to determine p_1 and λ_1, and also p_2 and λ_2 in terms of the former.

From the spherical triangle $C_1 f C_3$ (fig. 40*, Plate IV.*), we have by the formulæ of spherical trigonometry,

$$\cos fC_3 = \cos C_1C_3 \cos C_1f + \sin C_1C_3 \sin C_1f \cos fC_1C_3 ;$$

but the spherical angle fC_1C_3 is measured by the arc gC_3 at the equator.

Hence, substituting the values of these arcs given in the previous section, we have

$$\cos p_1 = \cos 90° \cos p_3 + \sin 90° \sin p_3 \cos \lambda_3$$
$$= \sin p_3 \cos \lambda_3 .$$

Again, in the spherical triangle fgC_3, we have

$$\frac{\sin fg}{\sin fC_3} = \frac{\sin f\,C_3\,g}{\sin fg\,C_3} ;$$

but spherical triangle fC_3g is measured by arc kC_2, and fgC_3 is $90°$; hence

$$\frac{\sin(90°-p_3)}{\sin p_1}=\frac{\sin\lambda_1}{\sin 90°} \quad \text{and } \sin\lambda_1=\frac{\cos p_3}{\sin p_1}$$

From the spherical triangle C_1C_2f, we have

$$\cos C_2f=\cos C_1C_2 \cos C_1f+\sin C_1C_2 \sin C_1f \cos C_2C_1f,$$

or, $\cos p_2=\cos 90° \cos p_3+\sin 90° \sin p_3 \cos(90°-\lambda_3)$
$=\sin p_3 \sin\lambda_3.$

From the spherical triangle C_2fg, we have

$$\frac{\sin C_2f}{\sin fg}=\frac{\sin C_2gf}{\sin fC_2g} \text{ or } \frac{\sin p_2}{\sin 90°-p_3}=\frac{\sin 90°}{\sin\lambda_2}$$

and $\sin\lambda_2=\dfrac{\cos p_3}{\sin p_2}$

Hence $\cos p_1=\sin p_3 \cos\lambda_3$, $\sin\lambda_1=\dfrac{\cos p_3}{\sin p_1}$;

and $\cos p_2=\sin p_3 \sin\lambda_3$, $\sin\lambda_2=\dfrac{\cos p_3}{\sin p_2}$

119. To find the angle between the poles of two faces in terms of their polar distances and longitudes.

Let C_1F be the polar distance of F (fig. 41, Plate IV.*), C_3L its longitude, C_1f the polar distance, and C_3l the longitude of f.

Also let $C_1F=P_3$, $C_1f=p_3$; $C_3L=L_3$, $C_3l=\lambda_3$, and $Ff=\theta$.

Then in spherical triangle C_1Ff

$$\cos Ff=\cos C_1F \cos C_1f+\sin C_1F \sin C_1f \sin FC_1f.$$

Then angle FC_1f is measured by arc $Ll=LC_3-lC_3$.

Hence $\cos\theta=\cos P_3 \cos p_3+\sin P_3 \sin p_3 \cos(L_3-\lambda_3)$.

To adapt this to logarithmic computation—

$$\cos\theta=\cos p_3\,\{\cos P_3+\sin P_3 \tan p_3 \cos(L_3-\lambda_3)\}.$$

Let $\tan a=\tan p_3 \cos(L_3-\lambda_3)$.

Then $\cos\theta=\cos p_3\,\{\cos P_3+\tan a\,.\,\sin P_3\}$

$$=\frac{\cos p_3}{\cos a}\{\cos P_3 \cos a+\sin P_3 \sin a\}$$

$$=\frac{\cos p_3}{\cos a}\cos(P_3-a).$$

120. To find the distance between any two poles on the sphere of projection in terms of the three polar distances from C_1, C_2, and C_3.

§ 119. $\cos\theta=\cos P_3 \cos p_3+\sin P_3 \sin p_3 \cos(L_3-\lambda_3)$
$=\cos P_3 \cos p_3+\sin P_3 \sin p_3 (\cos L_3 \cos\lambda_3$
$+\sin L_3 \sin\lambda_3)$
$=\cos P_3 \cos p_3+\sin P_3 \sin p_3 \cos L_3 \cos\lambda_3$
$+\sin P_3 \sin p_3 \sin L_3 \sin\lambda_3$;

but § 118, $\cos p_1=\sin p_3 \cos\lambda_3$ $\quad \cos P_1=\sin P_3 \cos L_3$
$\cos p_2=\sin p_3 \sin\lambda_3$ $\quad \cos P_2=\sin P_3 \sin L_3$.

Hence $\cos\theta = \cos P_3 \cos p_3 + \cos P_2 \cos p_2 + \cos P_1 \cos p_1$.

The same formulæ which we obtained by geometry of three dimensions, § 109.

121. To find the polar distances and longitudes in terms of the indices.

Referring to § 117 and (fig. 40*, Plate IV.*), C_1, C_2, and C_3 are poles of the cube, f is a pole of $1\,m\,n$, g of $1\,m\,\infty$, h of $1\,\infty\,n$, k of $\infty\,1\,\frac{n}{m}$, C_3 of $1\,\infty\,\infty$, C_2 of $\infty\,1\,\infty$, and C_1 of $\infty\,\infty\,1$. $fC_3 = p_1$, $fC_2 = p_2$, $fC_1 = p_3$, $C_2k = \lambda_1$, $C_3h = \lambda_2$, $C_3g = \lambda_3$.

Then λ_1 is the distance between the poles of $\infty\,1\,\frac{n}{m}$ and $\infty\,1\,\infty$, p_1 that between $1\,m\,n$ and $1\,\infty\,\infty$.

Hence, § 107,

$$\cos\lambda_1 = \frac{1}{\sqrt{1+\frac{m^2}{n^2}}} \qquad \cos p_1 = \frac{1}{\sqrt{1+\frac{1}{m^2}+\frac{1}{n^2}}}$$

$$\sec^2\lambda_1 = 1+\frac{m^2}{n^2} \qquad \sec^2 p_1 = 1+\frac{1}{m^2}+\frac{1}{n^2}$$

$$\tan^2\lambda_1 = \frac{m^2}{n^2} \qquad \tan^2 p_1 = \frac{1}{m^2}+\frac{1}{n^2} = \frac{1}{m^2}\left(1+\frac{m^2}{n^2}\right)$$

$$\tan\lambda_1 = \frac{m}{n} \qquad = \frac{1}{m^2}\sec^2\lambda_1$$

$$n = m\cot\lambda_1 \qquad \tan p_1 = \frac{1}{m}\sec\lambda_1$$

$$= \cot\lambda_1 \sec\lambda_1 \cot p_1 \qquad m = \sec\lambda_1 \cot p_1$$

$$= \frac{\cot p_1}{\sin\lambda_1}$$

Again λ_2 is the distance between $1\,\infty\,n$ and $1\,\infty\,\infty$, p_2 that between $1\,m\,n$ and $\infty\,1\,\infty$.

Hence, § 107,

$$\cos\lambda_2 = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \qquad \cos p_2 = \frac{\frac{1}{m}}{\sqrt{1+\frac{1}{m^2}+\frac{1}{n^2}}}$$

$$\sec^2\lambda_2 = 1+\frac{1}{n^2} \qquad \sec^2 p_2 = m^2\left(1+\frac{1}{m^2}+\frac{1}{n^2}\right)$$

$$\tan^2\lambda_2 = \frac{1}{n^2} \qquad = 1+m^2\left(1+\frac{1}{n^2}\right)$$

$$n = \cot\lambda_2 \qquad \tan^2 p_2 = m^2\sec^2\lambda_2$$

$$\tan p = m\sec\lambda_2$$

$$m = \tan p_2 \cos\lambda_2$$

Also λ_3 is the distance between $1\,m\,\infty$ and $1\,\infty\,\infty$, p_3 that between $1\,m\,n$ and $\infty\,\infty\,1$.

And, § 107,

$$\cos\lambda_3=\frac{1}{\sqrt{1+\frac{1}{m^2}}}\qquad \cos p_3=\frac{\frac{1}{n}}{\sqrt{1+\frac{1}{m^2}+\frac{1}{n^2}}}$$

$$\sec^2\lambda_3=1+\frac{1}{m^2}\qquad \sec^2 p_3=n^2\left(1+\frac{1}{m^2}+\frac{1}{n^2}\right)$$

$$\tan^2\lambda_3=\frac{1}{m^2}\qquad =1+n^2\left(1+\frac{1}{m^2}\right)$$

$$m=\cot\lambda_3\qquad \tan^2 p_3=n^2\sec^2\lambda_3$$

$$n=\tan p_3\cos\lambda_3.$$

Hence the indices being given, the polar distances and longitudes can be determined, or the polar distances and longitudes being given the indices can be determined.

122. To find the polar distances of any two adjacent poles of faces of the six-faced octahedron, or of the supplement of the angle over the edge of any two adjacent faces, in terms of the indices.

Let θ be the angle between any two poles adjacent to the arc CO (figs. 31* and 32*, Plate IV.*), ϕ adjacent to OD, and ψ adjacent to $C\,D$.

For the faces $n\,m\,1$, $m\,n\,1$,

$$\cos\theta=\frac{\frac{1}{mn}+\frac{1}{mn}+1}{\sqrt{\left(\frac{1}{n^2}+\frac{1}{m^2}+1\right)\left(\frac{1}{m^2}+\frac{1}{n^2}+1\right)}}=\frac{\frac{2}{mn}+1}{1+\frac{1}{m^2}+\frac{1}{n^2}}$$

Similarly for $1\,n\,m$ and $1\,m\,n$ we have

$$\cos\theta=\frac{\frac{2}{mn}+1}{1+\frac{1}{m^2}+\frac{1}{n^2}}$$

The same is true over every arc CO in (figs. 31* and 32*, Plate IV.*).

For the faces $m\,1\,n$ and $1\,m\,n$ $\cos\phi=\dfrac{\frac{1}{m}+\frac{1}{m}+\frac{1}{n^2}}{1+\frac{1}{m^2}+\frac{1}{n^2}}=\dfrac{\frac{2}{m}+\frac{1}{n^2}}{1+\frac{1}{m^2}+\frac{1}{n^2}}$

For the faces $m\,1\,n$, $m\,1\,\bar{n}$ $\cos\psi=\dfrac{\frac{1}{m^2}+1-\frac{1}{n^2}}{1+\frac{1}{m^2}+\frac{1}{n^2}}$

123. To express θ, ϕ, and ψ in terms of the polar distances and longitudes.

Then, according to § 110, if we substitute $\cos p_1$ for 1, $\cos p_2$ for m, and $\cos p_3$ for n,

We have for the faces $n\,m\,1$ and $m\,n\,1$, or

$$\cos p_3, \cos p_2, \cos p_1, \text{ and } \cos p_2, \cos p_3, \cos p_1.$$

and $\cos\theta = \cos p_2 \cos p_3 + \cos p_2 \cos p_3 + \cos^2 p_1$
$= 2\cos p_2 \cos p_3 + \cos^2 p_1.$

For the faces $m\,1\,n$, and $1\,m\,n$, or

$$\cos p_2, \cos p_1, \cos p_3, \text{ and } \cos p_1, \cos p_2, \cos p_3,$$

and $\cos\phi = \cos p_1 \cos p_2 + \cos p_1 \cos p_2 + \cos^2 p_3$
$= 2\cos p_1 \cos p_2 + \cos^2 p_3.$

Also for the faces $m\,1\,n$ and $m\,1\,\bar{n}$, or

$$\cos p_2, \cos p_1, \cos p_3, \text{ and } \cos p_2, \cos p_1, \cos p_3,$$

$\cos\psi = \cos^2 p_2 + \cos^2 p_1 - \cos^2 p_3.$

But referring to § 118 $\cos^2 p_2 = \sin^2 p_3 \sin^2 \lambda_3$
and $\cos^2 p_1 = \sin^2 p_3 \cos^2 \lambda_2.$

Hence $\cos\psi = \sin^2 p_3 \sin^2 \lambda_3 + \sin^2 p_3 \cos^2 \lambda_3 - \cos^2 p_3$
$= \sin^2 p_3 - \cos^2 p_3 = 2\sin^2 p_3 - 1.$

And $1 + \cos\psi = 2\sin^2 p_3.$

Therefore $2\cos^2 \frac{\psi}{2} = 2\sin^2 p_3,$

$$\text{and} \quad \cos\frac{\psi}{2} = \sin p_3 = \cos(90° - p_3).$$

Whence $\frac{\psi}{2} = 90° - p_3$, or $\psi = 180° - 2p_3.$

This result might have been obtained at once by inspection from (fig. 31*, Plate IV.*) For p_3 is the north polar distance of the face $1\,m\,n$, and $180° - p_3$ that of $1\,m\,\bar{n}$. The poles of both these faces also lie in the same meridian.

Hence $\phi = 180° - p_3 - p_3 = 180° - 2p_3.$

Again, using the formulæ § 119, θ is the inclination of the pole of the face $m\,n\,1$ to that of $n\,m\,1$, p_1 the north polar distance of the pole of $m\,n\,1$, and λ_1 its longitude referred to C_1 as north pole, and $C_3C_2C_5$ as equator and measured from C_3.

p_1 the north polar distance of $n\,m\,1$ and $90 - \lambda_1$ its longitude referred to the same north pole and equator.

Hence $\cos\theta = \cos p_1 \cos p_1 + \sin p_1 \sin p_1 \cos(90 - 2\lambda_1)$
$= \cos^2 p_1 + \sin^2 p_1 \cos(90 - 2\lambda_1)$
$= 1 - \sin^2 p_1 + \sin^2 p_1 \cos(90 - 2\lambda_1);$
and $1 - \cos\theta = \sin^2 p_1 \{1 - \cos(90 - 2\lambda_1)\}.$

Therefore $2\sin^2\frac{\theta}{2} = 2\sin^2 p_1 \sin^2 \frac{90 - 2\lambda_1}{2}$

$$\text{and} \quad \sin\frac{\theta}{2} = \sin p_1 \sin(45 - \lambda_1).$$

In like manner, since p_3 and λ_3; and p_3 and $90-\lambda_3$, are the polar distances and longitudes of the faces $1\,m\,n$ and $m\,1\,n$ referred to C_1 as north pole, and $C_3C_2C_5$ as equator,

$$\cos\phi=\cos p_3\cos p_3+\sin p_3\sin p_3\cos(90-2\lambda_3),$$

which gives as above

$$\sin\frac{\phi}{2}=\sin p_3\sin(45-\lambda_3).$$

124. Given ϕ and ψ, find p_3 and λ_3.

We have seen, § 123, that $p_3=90-\frac{\psi}{2}$;

$$\text{also }\sin\frac{\phi}{2}=\sin p_3\sin(45-\lambda_3),$$

$$\text{therefore}\quad\sin(45-\lambda_3)=\frac{\sin\frac{\phi}{2}}{\sin p_3}$$

125. Given ψ and θ, find p_3 and λ_3.

§ 123. $p_3=90-\frac{\psi}{2}$.

$$\sin\frac{\theta}{2}=\sin(45-\lambda_1)\sin p_1.$$

$$=(\sin 45\cos\lambda_1-\cos 45\sin\lambda_1)\sin p_1;$$

$$\text{but }\sin 45=\cos 45=\frac{1}{\sqrt{2}}$$

$$\therefore\ \sqrt{2}\ \sin\frac{\theta}{2}=\sin p_1\cos\lambda_1-\sin p_1\sin\lambda_1.$$

Referring to (fig. 40*, Plate IV.*), and remembering from § 117, that

$$\begin{array}{lll} p_1=fC_3 & p_2=fC_2 & p_3=fC_1 \\ \lambda_1=C_2k & \lambda_2=C_3h & \lambda_3=C_3g. \end{array}$$

From the spherical triangle $fg\,C_3$, we have

$$\frac{\sin fC_3}{\sin fg\,C_3}=\frac{\sin fg}{\sin f\,C_3g}\quad\text{or}\quad\frac{\sin p_1}{\sin 90}=\frac{\sin(90-p_3)}{\sin\lambda_1}=\frac{\cos p_3}{\sin\lambda_1}$$

Therefore $\sin p_1\sin\lambda_1=\cos p_3$.

Also from spherical triangle C_1fC_3, we have

$$\frac{\sin fC_3}{\sin fC_1C_3}=\frac{\sin fC_1}{\sin fC_3C_1}\quad\text{or}\quad\frac{\sin p_1}{\sin\lambda_3}=\frac{\sin p_3}{\sin(90-\lambda_1)}=\frac{\sin p_3}{\cos\lambda_1}.$$

Therefore $\sin p_1\cos\lambda_1=\sin p_3\sin\lambda_3$.

Hence

$$\sqrt{2}\sin\frac{\theta}{2}=\sin p_1\cos\lambda_1-\sin p_1\sin\lambda_1=\sin p_3\sin\lambda_3-\cos p_3$$

$$=\sin\left(90-\frac{\psi}{2}\right)\sin\lambda_3-\cos\left(90-\frac{\psi}{2}\right)=\cos\frac{\psi}{2}\sin\lambda_3-\sin\frac{\psi}{2}$$

$$\text{Hence }\cos\frac{\psi}{2}\sin\lambda_3=\sqrt{2}\sin\frac{\theta}{2}+\sin\frac{\psi}{2}=\sec 45^\circ\sin\frac{\theta}{2}+\sin\frac{\psi}{2},$$

$$\text{and } \sin\lambda_3 = \frac{1}{\cos\frac{\psi}{2}}\left\{\sec 45^\circ \sin\frac{\theta}{2} + \sin\frac{\psi}{2}\right\}$$

$$= \frac{\sec 45^\circ \sin\frac{\theta}{2}}{\cos\frac{\psi}{2}}\left\{1+\frac{\sin\frac{\psi}{2}}{\sec 45 \sin\frac{\theta}{2}}\right\}$$

$$\text{Let } \tan^2 a = \frac{\sin\frac{\psi}{2}\cos 45}{\sin\frac{\theta}{2}}$$

$$\text{Then } \sin\lambda_3 = \frac{\sin\frac{\theta}{2}}{\cos 45 \cos\frac{\psi}{2}}\{1+\tan^2 a\} = \frac{\sin\frac{\theta}{2}}{\cos 45 \cos\frac{\psi}{2}\cos^2 a}$$

126. Given ϕ and θ, find p_3 and λ_3.

(Fig. 42*, Plate IV*.)—Let a_1 be the pole of $1\,m\,n$, a_2 that of $1\,n\,m$, and a_3 that of $m\,1\,n$.

Join a_1, a_2 by arc of great circle cutting oC_3 in f,
and a_1, a_3 by arc of great circle cutting od in e;
also C_1, a_1 by $C_1\,a_1$ cutting dC_3 in g,
and Oa_1 cutting dC_3 in h.

Then $C_1a_1 = p_3$, $C_3g = \lambda_3$, $C_1o = 54^\circ\ 44'$, and $C_2od = 60^\circ$; and let $oa_1 = P$, $C_2oa_1 = L$.

Also $a_3a_1 = \phi$ and $ea_1 = \frac{\phi}{2}$ $\quad a_1a_2 = \theta$ $\quad a_1f = \frac{\theta}{2}$

From spherical triangle oa_1f $\quad \dfrac{\sin a_1f}{\sin a_1of} = \dfrac{\sin oa_1}{\sin ofa_1}$

$$\text{therefore } \frac{\sin\frac{\theta}{2}}{\sin L} = \frac{\sin P}{\sin 90^\circ}$$

Also in spherical triangle oa_1e $\quad \dfrac{\sin a_1e}{\sin eoa_1} = \dfrac{\sin oa_1}{\sin oea_1}$

$$\text{and } \frac{\sin\frac{\phi}{2}}{\sin(60^\circ - L)} = \frac{\sin P}{\sin 90^\circ}$$

$$\text{Hence } \frac{\sin\frac{\theta}{2}}{\sin L} = \frac{\sin\frac{\phi}{2}}{\sin(60^\circ - \mathrm{L})}$$

$$\text{and } \frac{\sin\frac{\theta}{2}}{\sin\frac{\phi}{2}} = \frac{\sin L}{\sin(60^\circ - L)} = \frac{\sin L}{\sin 60^\circ \cos L - \cos 60^\circ \sin L}$$

$$\frac{\sin\frac{\theta}{2}}{\sin\frac{\phi}{2}}=\frac{\sin L}{\frac{\sqrt{3}}{2}\cos L-\frac{1}{2}\sin L}$$

$$\frac{\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}=\frac{\sqrt{3}}{2}\cot L-\frac{1}{2}$$

$$\sqrt{3}\cot L=1+\frac{2\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}$$

$$\text{Let } \tan^2 a=\frac{2\sin\frac{\phi}{2}}{\sin\frac{\theta}{2}}=\frac{\sin\frac{\phi}{2}}{\sin 30^\circ\sin\frac{\theta}{2}}$$

Therefore $\sqrt{3}\cot L=1+\tan^2 a=\dfrac{1}{\cos^2 a}$

and $\tan L=\sqrt{3}\cos^2 a=\tan 60^\circ\cos^2 a$.

But we have seen that $\dfrac{\sin P}{\sin 90^\circ}=\dfrac{\sin\frac{\theta}{2}}{\sin L}$ and $\sin P=\dfrac{\sin\frac{\theta}{2}}{\sin L}$

Also from spherical triangle C_1oa_1 we have

$$\cos C_1a_1=\cos C_1o\cos oa_1+\sin C_1o\sin oa_1\cos C_1oa_1;$$

$$\begin{aligned}\text{or } \cos p_3&=\cos 54^\circ 44'\cos P+\sin 54^\circ 44'\sin P\cos(120^\circ+L).\\&=\cos 54^\circ 44'\cos P-\sin 54^\circ 44'\sin P\cos(60^\circ-L).\\&=\cos P\,[\cos 54^\circ 44'-\sin 54^\circ 44'\tan P\cos(60^\circ-L)].\end{aligned}$$

Let $\tan\beta=\tan P\cos(60^\circ-L)$.

$$\begin{aligned}\text{Therefore } \cos p_3&=\cos P\,\{\cos 54^\circ 44'-\sin 54^\circ 44'\tan\beta\}\\&=\frac{\cos P}{\cos\beta}\{\cos 54^\circ 44'\cos\beta-\sin 54^\circ 44'\sin\beta\}\end{aligned}$$

$$\cos p_3=\frac{\cos P}{\cos\beta}\cos(54^\circ 44'+\beta).$$

Also in spherical triangle C_1ea_1,

$$\frac{\sin C_1a_1}{\sin C_1ea_1}=\frac{\sin a_1e}{\sin eC_1a_1}\quad\text{or}\quad\frac{\sin p_3}{\sin 90^\circ}=\frac{\sin\frac{\phi}{2}}{\sin(45^\circ-\lambda_3)}$$

$$\text{and } \sin(45^\circ-\lambda_3)=\frac{\sin\frac{\phi}{2}}{\sin p_3}$$

Hence on the whole we have the formulæ

$$\tan^2 a = \frac{\sin \frac{\phi}{2}}{\sin 30^\circ \sin \frac{\theta}{2}}$$

$$\tan L = \tan 60^\circ \cos^2 a.$$

$$\sin P = \frac{\sin \frac{\theta}{2}}{\sin L} \quad \tan \beta = \tan P \cos (60^\circ - L).$$

$$\cos p_3 = \frac{\cos P}{\cos \beta} \quad \cos (54^\circ\ 44' + \beta).$$

and $\sin (45^\circ - \lambda_3) = \dfrac{\sin \frac{\phi}{2}}{\sin p_3}$

for determining p_3 and λ_3 in terms of ϕ and θ; all the formulæ being adapted for logarithmic computation.

p_3 and λ_3 being determined from the values of ϕ, θ, and ψ, m and n can be expressed in terms of p_3 and λ_3.

127. By the formulæ given in § 124, § 125, and § 126, any two of the angles of inclination such as ϕ, θ, and ψ, over the edges of a six-faced octahedron, having been observed by the goniometer, p_3 and λ_3 can be determined. Again, by formulæ in § 118, p_1 and λ_1, p_2 and λ_2 can be obtained from the values of p_3 and λ_3.

p_3 and λ_3 being determined, m and n can be obtained. Now all the forms of the cubical system are derived from those of the six-faced octahedron.

Hence by determining θ, ϕ, and ψ for any form of the cubical system, we can obtain the values both of p_3 and λ_3, and also of the indices 1, m, and n.

As we advance in this treatise we shall show good reasons for preferring the polar circular co-ordinates p_3 and λ_3 to the linear ratios or fractions m and n.

128. The problems of crystallography being resolved for the most part into those of spherical trigonometry, may be solved by means of lines drawn on the surface of a solid sphere.

This being inconvenient in practice, it is usual to project the points or poles on the surface of the sphere upon those of a plane, just as geographical and astronomical maps are projections from the surface of the sphere upon the plane of the paper on which the map is drawn. There are three principal projections of the sphere,—the *steregraphic*, *orthographic*, and *gnomic*.

The *steregraphic* when the eye is supposed to be placed on the surface of the sphere and the points in the hemisphere furthest from the eye are projected on the plane of the equator;

considering the point of sight or projection, the pole of the great circle on which the projection is made.

In this projection the projections of circles on the sphere are either straight lines or circles.

The *orthographic* where the eye is supposed to be placed at an infinite distance from the sphere. In this projection points on the surface of the sphere are projected on the plane of the equator by perpendiculars from those points to that plane.

In this case all great circles inclined to the equator are projected into ellipses on the plane of projection.

The *gnomic* where the eye is placed in the centre of the sphere, and the plane of projection is a plane touching the surface of the sphere.

In this projection all great circles are projected into a straight line.

From the difficulty of describing arcs of ellipses the *orthographic* projection is not suited to crystallographical problems.

The *steregraphic* is that mostly used by Professor Miller and other distinguished crystallographers, but there is some trouble in finding the centres of the arcs of great circles on the sphere of projection.

The most simple projection for most purposes is the *gnomic*. By either the steregraphic or gnomic projection, many problems may be very expeditiously solved by simple geometrical constructions.

129. Comparing (fig. 14, Plate II.) with (fig. 27, Plate IV.), we see that if we take A, the centre of the cube, for the centre of the sphere of projection, and Ao_1, Ao_2, &c., Ao_8 as equal radii of that sphere,—the eight faces, C_1, C_2, C_3, &c., of the octahedron will each be tangent planes, touching the sphere in the eight points o_1, o_2, &c., o_8. Because each of these plane faces are respectively perpendicular to Ao_1, Ao_2, &c., at the points o_1, o_2, &c.

The projections on the faces of the octahedron will be the same as in the former case if we regard the sphere of projection as the sphere inscribed in the cube touching the cube in the points C_1, C_2, &c., C_6.

All the poles, therefore, of all the forms of the cubical system can therefore be projected on to the planes of the octahedron inscribed in the cube,—one octant of the sphere upon each face. In (fig. 14, Plate II.), as shown in perspective, and (fig. 33, Plate IV.), on the plane of the paper,—the equilateral triangle $C_1C_2C_3$ represents the gnomic projection of an octant of the sphere of projection.

$C_1C_2C_3$ being the projections of three poles of the cube.

Bisect C_1C_2 in d_1, C_1C_3 in d_2, and C_2C_3 in d_5.

d_1, d_2, and d_5 are projections of the poles of three faces of the rhombic dodecahedron.

Join C_1d_5, C_2d_2, and C_3d_1 meeting in o_1; o_1 is the projection of the pole of a face of the octahedron.

(Fig. 43*, Plate IV.*)—$\mathbf{B}_1$, $\mathbf{E}_1$, $\mathbf{F}_1$, $\mathbf{G}_1$, $\mathbf{H}_1$, $\mathbf{K}_1$, $\mathbf{L}_1$, $\mathbf{M}_1$, $\mathbf{N}_1$, $\mathbf{P}_1$, $\mathbf{Q}_1$ represent the poles of nearly all the known four-faced cubes lying in the arc of the zone d_5C_3; $\mathbf{B}_2$, &c., in C_3d_2; $\mathbf{B}_3$, &c., in C_1d_2; $\mathbf{B}_4$, &c., in C_2d_5; $\mathbf{B}_5$, &c., in C_2d_5; and $\mathbf{B}_6$, &c., in C_1d_5. Six poles of each four-faced cube in the octant at equal distances from C_1, C_2, and C_3.

Rules for finding the position of $\mathbf{B}_1$, $\mathbf{E}_1$, &c., will be given hereafter.

b_1, e_1, f_1, g_1, h_1, k_1, and l_1; b_2, e_2, &c., l_2; and b_3, e_3, &c., l_3, three poles of each three-faced octahedron, lying at equal distances from o_1, in the arcs of zones represented respectively by o_1d_5, o_1d_2, and o_1d_1.

b_1, e_1, f_1, g_1, h_1, k_1, l_1, m_1, n_1, o_1, p_1, and q_1; b_2, e_2, &c., q_2; b_3, e_3, &c., q_3, three poles of each twenty-four-faced trapezohedron, lying at equal distances from o_1, in arcs of zones represented by o_1C_3, o_1C_1, and o_1C_2 respectively.

Lastly A_1, B_1, E_1, F_1, G_1, H_1, K_1, L_1, M_1, N_1, P_1, Q_1, R_1, S_1, T_1, U_1; A_2, B_2, &c., U_2; A_3, B_3, &c., U_3; A_4, B_4, &c., U_4; A_5, B_5, &c., U_5; and A_6, B_6, &c., U_6, six poles of the six-faced octahedron; the poles of each particular six-faced octahedron being similarly situated in each of the six triangles $d_5o_1C_3$, $d_2o_1C_3$, $d_2o_1C_1$, $d_5o_1C_2$, $d_1o_1C_2$, and $d_1o_1C_1$ respectively.

130. To find geometrically the position of any pole on the gnomic projection (fig. 43*, Plate IV.*).

In (fig. 44*, Plate IV.*).—Let AC_3, AC_2, and AC_1 be three adjacent cubical axes, rectangular at A.

Let $AC_3=1$. Take AN in AC_1 produced equal to n.

AM in AC_2 produced equal to m.

Join C_3N, NM, MC_3, C_3C_2, C_2C_1, and C_1C_3.

Then C_2MN is the plane $1\,m\,n$, and $C_1C_3C_2$ is the plane of the gnomic projection.

Through A draw AG perpendicular, C_3M meeting C_2C_3 in g, AH perpendicular C_3N, cutting C_1C_3 in h, and AK perpendicular to C_1C_2, cutting C_1C_2 in k.

h, g, and k are the projections on $C_1C_2C_3$ of H, G, and K. Join NG, MH, and C_3K in the plane NMC_3, meeting in F; also join AF. Then, as in § 117, F is the pole of $1\,m\,n$, G of $1\,m\,\infty$, H of $1\,\infty\,n$, and K of $\infty\,1\,\frac{n}{m}$.

Therefore on the plane of projection, $C_1C_2C_3$, g is the projection of the pole of $1\,m\,\infty$, h of $1\,\infty\,n$, and k of $\infty\,1\,\frac{n}{m}$; hC_2 of the line HM, kC_3 of the line KC_3, gC_1 of the line GN.

f, where hC_2, kC_3, and gC_1 meet, will be the pole of $1\,m\,n$. Through h, in the plane NAC_3, draw hE perpendicular to AC_3.

Let angle $hAC_3=\lambda_2$. Then since angle $AHN=90°$, angle $ANH=\lambda_2$.

In triangle NAC_3 $\tan ANC_3=\frac{AC_3}{AN}$ or $\tan \lambda_2=\frac{1}{n}$

In triangle AhE $\tan hAE=\tan \lambda_2=\frac{hE}{AE}$

Hence $\frac{hE}{AE}=\frac{1}{n}$ and $hE=\frac{AE}{n}=\frac{AC_3-2C_3}{n}=\frac{1-EC_3}{n}$

But by similar triangles hEC_3, C_1AC_3,

$\frac{hE}{EC_3}=\frac{C_1A}{AC_3}=\frac{1}{1}$. Therefore $hE=EC_3$;

and $EC_3=\frac{1-EC_3}{n}$ and $nEC_3=1-EC_3$.

Whence $EC_3=\frac{1}{n+1}$

But by similar triangles C_1AC_3, hEC_3,

$\frac{AC_3}{EC_3}=\frac{C_1C_3}{hC_3}$ but $AC_3=1$ and $EC_3=\frac{1}{n+1}$

Hence $\frac{C_1C_3}{hC_3}=n+1$, and $C_3h=\frac{C_1C_3}{n+1}$

Hence h, the pole of $1\,\infty\,n$, is found by taking the point h in C_1C_3, so that $C_3h=\frac{C_1C_3}{n+1}$

Again since $\tan \lambda_2=\frac{1}{n}$ and angle $hAC_3=\lambda_2$, if the angular elements be given, C_1C_3 is the chord of 90° and h is the point where the angle λ_2 protracted from A meets C_1C_3, considering C_3 as zero.

The chord of 90° marked as a protractor is obtainable from any mathematical instrument maker, or may be readily marked on the chord of 90° by using any form of protractor.

Similarly it may be shown that $gC_3=\frac{C_2C_3}{m+1}$, and that g is the point where the angle λ_3 is marked on C_2C_3 as the chord of 90°, C_3 being zero; and $\tan \lambda_3=\frac{1}{m}$. Also $kC_2=\frac{C_1C_2}{\frac{m}{n}+1}$, k being the point where the angle λ_1 is marked on the chord of 90°, C_2 being zero, and $\tan \lambda_1=\frac{m}{n}$.

Join C_1g, C_2h, and C_3k. f, the point where these three lines meet, is the pole of the face of the six-faced octahedron whose angular elements are p_3 and λ_3, or whose indices are $1\,m\,n$.

131. To construct a map of all the forms of the octahedral system on a face of an octahedron comprised in an octant of the sphere of projection.

(Fig. 43*, Plate IV.*) Describe any equilateral triangle $C_1C_2C_3$.

Bisect C_1C_2 in d_1, C_1C_3 in d_2, and C_2C_3 in d_5.

Then C_3 is the pole of $1\,\infty\,\infty$, C_2 of $\infty\,1\,\infty$, and C_1 of $\infty\,\infty\,1$, three poles of the cube.

d_1 is the pole of $\infty\,1\,1$, d_2 of $1\,\infty\,1$, and d_5 of $1\,1\,\infty$, three poles of the rhombic dodecahedron.

Join C_1d_5, C_2d_2, and C_3d_1 meeting in o. Then o is the pole of the face of the octahedron whose symbol is $1\,1\,1$.

To place on this octant six poles of the six-faced octahedron whose indices are $1, \frac{4}{3}, 2$.

In this case $\lambda_3=36°\ 52'$, $\lambda_2=26°\ 34'$, and $\lambda_1=33°\ 41'$.

Graduate each of the lines C_3d_2, C_3d_5, C_1d_2, C_1d_1, C_2d_1, and C_2d_5, from $o°$ to $45°$; considering C_1C_2, C_2C_3 and C_1C_3 as chords of $90°$, and making the three points C_1, C_2, C_3 each zero, as described in § 132.

Let $C_3\mathbf{F}_1=36°\ 52'=C_3\mathbf{F}_2=C_1\mathbf{F}_3=C_2\mathbf{F}_4=C_2\mathbf{F}_5=C_1\mathbf{F}_6$

$C_3\mathbf{H}_1=26°\ 34'=C_3\mathbf{H}_2=C_1\mathbf{H}_3=C_2\mathbf{H}_4=C_2\mathbf{H}_5=C_1\mathbf{H}_6$

$C_3\mathbf{G}_1=33°\ 41'=C_3\mathbf{G}_2=C_1\mathbf{G}_3=C_1\mathbf{G}_4=C_2\mathbf{G}_5=C_1\mathbf{G}_6$

Then E_1 is the intersection of $C_1\mathbf{F}_1$, $C_2\mathbf{H}_2$, $C_3\mathbf{G}_5$
E_2 ,, of $C_1\mathbf{H}_1$, $C_2\mathbf{F}_2$, $C_3\mathbf{G}_6$
E_3 ,, of $C_1\mathbf{G}_1$, $C_2\mathbf{F}_3$, $C_3\mathbf{H}_6$
E_4 ,, of $C_1\mathbf{F}_4$, $C_2\mathbf{G}_2$, $C_3\mathbf{H}_5$
E_5 ,, of $C_1\mathbf{H}_4$, $C_2\mathbf{G}_3$, $C_3\mathbf{F}_5$
E_6 ,, of $C_1\mathbf{G}_4$, $C_2\mathbf{H}_3$, $C_3\mathbf{F}_6$

E_1, E_2, E_3, E_4, E_5, and E_6 will be six poles of the six-faced octahedron whose indices are $1, \frac{4}{3}, 2$, and angular elements $\lambda_3=36°\ 52'$, $p_3=68°\ 12'$. The lines of intersection are not shown in the plate.

(Fig. 43*, Plate IV.*) has marked on it the poles on the octant of a sphere of nearly all the forms of the cubical system which have been observed; all the faces whose poles lie in the same line having their poles on the sphere of projection on the same zone circle.

The angular and linear indices of every form are given in the following table.

Where p_1, p_2, and p_3 are the polar distances of each form from the three poles of the poles of the cube, C_1, C_2, and C_3, θ, ϕ, and ψ the supplements of the angles of inclination over the edges of adjacent faces determined as in § 123, 124, 125, and 126.

§ 124, 125, and 126 show how when these angles or any two of them are determined from observation, the angular or linear elements can be determined from them.

The linear elements have hitherto been almost universally used as a concise means of expressing any form. Their disadvantages will be explained hereafter.

The angular elements are in reality more concise, because they can express the forms they represent to any degree of accuracy which can be derived from observation.

They have also this great advantage, that by the use of angles alone they can express the relations of any form to another without determining the linear elements at all.

Thus in the following table p_1 for any form gives the inclination of the face for which it stands to that of the adjacent face of the cube in any combination of these two forms.

Faces of all the twenty-four faced trapezohedrons lie in the same zone C_1od_5. Hence the value of p_1 for any of these faces gives the inclination of that face to that of the cube in that zone.

For instance (fig. 43*, Plate IV.*), m_2 is the pole of a face of the twenty-four-faced trapezohedron, for which the value of $p_3=78° 54'$, $\lambda_3=11° 19'$, linear elements 1, 5, 5; l_2 is the pole of another twenty-four-faced trapezohedron, where $p_3=76° 22'$, $\lambda_3=14° 2'$, linear elements 1, 4, 4.

For m_2; $p_1=15° 48'$. And for l_2; $p_1=19° 28'$.

Hence $54° 44'-15° 48'=Om_2$; $54° 44'-19° 28'=Ol_2$; and $19° 28'-15° 48'=m_2l_2$.

Results procured by simple subtraction when the angular elements are used; but only found by retranslating the linear indices obtained from angular observations of the goniometer back again into angles, by trigonometrical formulæ.

Again, referring to (fig. 43*, Plate IV.*), we see that C_1, U_3, Q_3, H_3, h_2, E_2, f_1, N_1, P_1, $\mathbf{H}_1$ all lie in the same meridional zone.

The values of p_1 for each of these forms enable us to determine the distances of these poles from each other in the zone by simple subtraction of angles.

Table of all the principal forms of the Cubical System.

SIX-FACED OCTAHEDRON.

	1 m n	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
A	1 $\frac{64}{63}$ 64	64 *O* $\frac{64}{63}$	64 63 1	44° 33′	0° 54′	0° 55′	89° 72′	45° 27′	44° 33′	58° 26′	0° 54′	1° 16′
B	1 $\frac{5}{4}$ $\frac{5}{3}$	$\frac{5}{3}$ *O* $\frac{5}{4}$	5 4 3	38 40	30 58	36 52	64 54	55 33	45 0	11 29	11 29	50 12
E	1 $\frac{4}{3}$ 2	2 *O* $\frac{4}{3}$	4 3 2	36 52	26 34	33 41	68 12	56 9	42 2	15 5	15 5	43 36
F	1 $\frac{4}{3}$ 4	$\frac{4}{3}$ *O* 4	4 3 1	36 52	14 2	18 26	78 41	53 58	38 20	32 12	15 57	22 37
G	1 $\frac{15}{11}$ $\frac{15}{7}$	$\frac{15}{7}$ *O* $\frac{15}{11}$	15 11 7	36 15	25 1	32 28	69 23	56 24	41 0	16 22	16 22	41 15
H	1 $\frac{3}{2}$ 3	3 *O* $\frac{3}{2}$	3 2 1	33 41	18 26	26 34	74 30	57 41	36 42	21 47	21 47	31 0
K	1 $\frac{8}{5}$ 8	8 *O* $\frac{8}{5}$	8 5 1	32 0	7 8	11 19	83 57	58 12	32 31	34 42	25 50	12 6
L	1 $\frac{5}{3}$ 5	5 *O* $\frac{5}{3}$	5 3 1	30 58	11 29	18 26	80 16	59 32	32 19	27 40	27 40	19 28
M	1 $\frac{5}{3}$ 10	10 *O* $\frac{5}{3}$	10 6 1	30 58	5 43	9 28	85 6	59 10	31 19	35 10	27 58	9 48
N	1 2 4	4 *O* 2	4 2 1	26 34	14 2	26 34	77 24	64 7	29 12	17 45	35 57	25 13
P	1 2 10	10 *O* 5	10 5 1	26 34	5 43	11 19	84 53	63 33	27 1	29 11	36 43	10 14
Q	1 $\frac{11}{5}$ $\frac{11}{3}$	$\frac{11}{3}$ *O* $\frac{11}{5}$	11 5 3	24 26	15 15	30 58	76 3	66 19	27 56	13 3	39 51	27 53
R	1 $\frac{16}{7}$ 4	4 *O* $\frac{16}{7}$	16 7 4	23 38	14 2	29 45	77 6	67 0	26 45	13 36	41 37	25 48
S	7 $\frac{7}{3}$ 7	7 *O* $\frac{7}{3}$	7 3 1	23 12	8 8	18 26	82 31	67 1	24 19	21 15	43 13	14 58
T	1 3 $\frac{21}{5}$	$\frac{21}{5}$ *O* 3	21 7 5	18 26	13 24	35 32	77 16	72 2	22 16	7 8	51 44	25 28
U	1 4 8	8 *O* 4	8 2 1	14 2	7 8	26 34	83 5	76 4	15 37	9 46	61 26	13 50

THREE-FACED OCTAHEDRON.

	1 m n	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
b	1 1 $\frac{65}{64}$	$\frac{65}{64}$ *O*	65 65 1	45° 0′	44° 33′	44° 33′	55° 9′	54° 32′	54° 32′	0° 43′	0° 0′	69° 42′
e	1 1 $\frac{5}{4}$	$\frac{5}{4}$ *O*	5 5 4	45 0	38 40	38 40	60 30	52 1	52 1	9 59	0 0	59 0
f	1 1 $\frac{3}{2}$	$\frac{3}{2}$ *O*	3 3 2	45 0	33 41	33 41	64 46	50 14	50 14	17 20	0 0	50 29
g	1 1 $\frac{7}{4}$	$\frac{7}{4}$ *O*	7 7 4	45 0	29 45	29 45	68 0	49 2	49 2	22 55	0 0	44 0
h	1 1 2	2 *O*	2 2 1	45 0	26 34	26 34	70 32	48 11	48 11	27 16	0 0	38 57
k	1 1 3	3 *O*	3 3 1	45 0	18 26	18 26	76 44	46 30	46 30	37 52	0 0	26 32
l	1 1 4	4 *O*	4 4 1	45 0	14 2	14 2	79 59	45 52	45 52	43 21	0 0	20 3

OCTAHEDRON.

	1 m n	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
O	1 1 1	*O*	1 1 1	45° 0′	45° 0′	45° 0′	54° 44′	54° 44′	54° 44′	0° 0′	0° 0′	70° 32′

TWENTY-FOUR-FACED TRAPEZOHEDRON.

	1 *m n*	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
b	1 $\frac{4}{3}$ $\frac{4}{3}$	$\frac{4}{3}$ *O* $\frac{4}{3}$	4 3 3	36° 52′	36° 52′	45° 0′	59° 1′	59° 1′	46° 39′	0° 0′	13° 56′	61° 56′
e	1 $\frac{3}{2}$ $\frac{3}{2}$	$\frac{3}{2}$ *O* $\frac{3}{2}$	3 2 2	33 41	33 41	45 0	60 59	60 59	43 19	0 0	19 45	58 2
f	1 2 2	2 *O* 2	2 1 1	26 34	26 34	45 0	65 54	65 54	35 16	0 0	33 33	48 11
g	1 $\frac{9}{4}$ $\frac{9}{4}$	$\frac{9}{4}$ *O* $\frac{9}{4}$	9 4 4	23 58	23 58	45 0	67 54	67 54	32 8	0 0	38 51	44 12
h	1 $\frac{8}{3}$ $\frac{8}{3}$	$\frac{8}{3}$ *O* $\frac{8}{3}$	8 3 3	20 33	20 33	45 0	70 39	70 39	27 56	0 0	45 58	38 42
k	1 3 3	3 *O* 3	3 1 1	18 26	18 26	45 0	72 27	72 27	25 14	0 0	50 29	35 6
l	1 4 4	4 *O* 4	4 1 1	14 2	14 2	45 0	76 22	76 22	19 28	0 0	60 0	27 16
m	1 5 5	5 *O* 5	5 1 1	11 19	11 19	45 0	78 54	78 54	15 48	0 0	65 57	22 11
n	1 10 10	10 *O* 10	10 1 1	5 43	5 43	45 0	84 19	84 19	8 3	0 0	78 7	11 22
o	1 12 12	12 *O* 12	12 1 1	4 46	4 46	45 0	85 15	85 15	6 43	0 0	80 8	9 30
p	1 16 16	16 *O* 16	16 1 1	3 25	3 25	45 0	86 26	86 26	5 3	0 0	82 39	7 8
q	1 40 40	40 *O* 40	40 1 1	1 26	1 26	45 0	88 34	88 34	2 2	0 0	87 6	2 52

FOUR-FACED CUBE.

	1 *m n*	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
B	1 $\frac{6}{5}$ ∞	∞ *O* $\frac{6}{5}$	6 5 0	39° 48′	0° 0′	0° 0′	90° 0′	50° 12′	39° 48′	53° 49′	10° 54′	0° 0′
E	1 $\frac{5}{4}$ ∞	∞ *O* $\frac{5}{4}$	5 4 0	38 40	0 0	0 0	90 0	51 20	38 40	52 26	12 40	0 0
F	1 $\frac{4}{3}$ ∞	∞ *O* $\frac{4}{3}$	4 3 0	36 52	0 0	0 0	90 0	53 8	36 52	50 12	16 16	0 0
G	1 $\frac{3}{2}$ ∞	∞ *O* $\frac{3}{2}$	3 2 0	33 41	0 0	0 0	90 0	56 19	33 41	46 11	22 38	0 0
H	1 2 ∞	∞ *O* 2	2 1 0	26 34	0 0	0 0	90 0	63 26	26 34	36 52	36 52	0 0
K	1 $\frac{7}{3}$ ∞	∞ *O* $\frac{7}{3}$	7 3 0	23 12	0 0	0 0	90 0	66 48	23 12	32 23	43 36	0 0
L	1 $\frac{5}{2}$ ∞	∞ *O* $\frac{5}{2}$	5 2 0	21 48	0 0	0 0	90 0	68 12	21 48	30 27	46 24	0 0
M	1 3 ∞	∞ *O* 3	3 1 0	18 26	0 0	0 0	90 0	71 34	18 26	25 51	53 8	0 0
N	1 4 ∞	∞ *O* 4	4 1 0	14 2	0 0	0 0	90 0	75 58	14 2	19 45	61 56	0 0
P	1 5 ∞	∞ *O* 5	5 1 0	11 19	0 0	0 0	90 0	78 41	11 19	15 57	67 22	0 0
Q	1 40 ∞	∞ *O* 40	40 1 0	1 26	0 0	0 0	90 0	88 34	1 26	2 56	87 8	0 0

RHOMBIC DODECAHEDRON.

	1 *m n*	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
d	1 1 ∞	∞ *O*	1 1 0	45° 0′	0° 0′	0° 0′	90° 0′	45° 0′	45° 0′	60° 0′	0° 0′	0° 0′

CUBE.

	1 *m n*	Naumann.	Miller.	λ_3	λ_2	λ_1	p_3	p_2	p_1	θ	ϕ	ψ
C	1 ∞ ∞	∞ *O* ∞	1 0 0	0° 0′	0° 0′	45° 0′	90° 0′	90° 0′	0° 0′	0° 0′	90° 0′	0° 0′

132. *Hemihedral or Half-symmetrical Forms of the Cubic System.*

In the holohedral or perfectly symmetrical forms of the cubical system, the solid form of the crystal is bounded by the lines where any one plane or face is intersected by the adjacent planes or faces. There are, however, symmetrical forms where half the number of the holohedral faces are omitted, the planes of the remaining faces forming a solid by the intersection of the adjacent planes.

These, called *hemihedral* or *half-symmetrical* faced forms, are of two kinds,—the *inclined,* in which no one face is parallel to the other; and the *parallel,* in which the faces are parallel in pairs.

133. The *inclined* hemihedral forms are the *tetrahedron* (figs. 15 and 16, Plate III.), the *twelve-faced trapezohedron* (figs. 17 and 18), the *four-faced tetrahedron* (figs. 19 and 20), and the *six-faced tetrahedron* (figs. 21 and 22); these being the hemihedral forms respectively derived from the *octahedron, three-faced octahedron, twenty-four-faced trapezohedron, and six-faced octahedron,* half of whose faces are produced to meet each other.

There are two hemihedral forms with *parallel* faces,—the *twelve-faced pentagon,* derived from the *four-faced cube* (figs. 23 and 24), and the *irregular twenty-four-faced trapezohedron,* derived from the *six-faced octahedron.*

The *cube* and *rhombic dodecahedron* do not produce *hemihedral* forms, according to the laws of symmetry by which the preceding are formed.

134. The *tetrahedron* (figs. 15 and 16, Plate III.) is formed by taking half the faces of the octahedron (fig. 7, Plate I.), in the following order,—$C_1C_2C_3$, $C_1C_5C_4$, $C_2C_5C_6$, and $C_4C_3C_6$, and producing these planes to intersect in the lines O_4O_2, O_2O_5, O_2O_7, O_4O_5, O_4O_7, and O_7O_5. Referring to (fig. 14, Plate II.), we see that these edges are diagonals of the square faces of the cube in which the octahedron is inscribed, one edge for each face of the cube.

The *tetrahedron* is therefore geometrically inscribed in the same cube in which the octahedron, from which it is derived, is also inscribed. (Fig. 16, Plate III.) shows the face of the octahedron shaded on the corresponding face of the tetrahedron.

Since O_2O_4, O_2O_5, and O_4O_5 are diagonals of equal squares, each face of the tetrahedron is an equilateral triangle, $O_2O_4O_5$ (fig. 33, Plate IV.). If we bisect the three sides of this equilateral triangle in the points C_1, C_2, and C_3, and join these points, the equilateral triangle $C_1C_2C_3$ will be a face of the octahedron.

If, therefore, we describe an equilateral triangle (fig. 33, Plate IV.), having each of its sides equal O_4O_5, (fig. 27, Plate IV.), four such triangles joined together will form the net of a tetrahedron which may be inscribed in the cube, each of whose faces equal the square $O_1O_4O_8O_5$ (fig. 27, Plate IV.).

Besides the tetrahedron just described, another in all respects similar and equal to the former, except as regards its position in the cube, may be formed by producing the four faces of the octahedron $C_1C_2C_5$, $C_1C_3C_4$, $C_2C_3C_6$, and $C_5C_4C_6$ (omitted in the former case), to meet each other. It is customary to call one of these *tetrahedrons* the *positive*, and the other the *negative*. Crystals of the following minerals have faces parallel to those of the tetrahedron :—

Blende (sulphuret of zinc), boracite, diamond, eulytine (bismuth blende), fahlerz (grey copper), pharmacosiderite (arseniate of iron), rhodizite, tennantite, and tritonite.

Naumann's symbol for the *tetrahedron* is $\frac{O}{2}$, Miller's $\kappa\,1\,1\,1$.

135. The *twelve-faced trapezohedron* is a half-symmetrical form with inclined faces derived from the *three-faced octahedron*, bounded by twelve equal and similar trapezohedrons (figs. 17 and 18, Plate III.). It is also called the *deltoidal dodecahedron*, the *trapezoidal dodecahedron*, and the *hemi-tri-octahedron*.

It is formed by producing the three faces of the three-faced octahedron corresponding to each face of the octahedron which are produced to form the tetrahedron, to form a solid by their intersection with each other.

Thus, comparing (figs. 17 and 18, Plate III.), with (fig. 6, Plate I.), the three faces meeting respectively in o_1, o_3, o_6, and o_8 of the three-faced octahedron, are produced to meet in the points W_2, W_4, W_5, and W_7, making, by their intersections, a twelve-faced trapezohedron bounded by twelve equal and similar trapeziums, $W_2C_1o_1C_3$, $W_4C_1o_1C_2$, &c.

If we call this the *positive twelve-faced trapezohedron*, the *negative* will be formed by the twelve faces of the three-faced octahedron which meet in groups of three in the points o_2, o_4, o_5, and o_7.

To obtain a face of the twelve-faced trapezohedron geometrically from the three-faced octahedron from which it is derived.

Describe the (fig. 29, Plate IV.), as previously shown in § 35, for determining the face of the three-faced octahedron. Produce C_1A to C_6, and O_1D_5 to O_5. Take $AC_6=D_5O_5=C_1A$. Join C_6O_5 and AO_5.

Produce Md_5 to meet AO_5 in W_5. Join C_6W_5.

Then (fig. 32, Plate IV.) $o_1 O_2 O_3$ being a face of the three-faced octahedron, bisect $O_2 O_3$ in d_5. Join $o_1 d_5$, and produce it to W_5, making $o_1 d_5 W_5 = o_1 d_5 W_5$ (fig. 29, Plate IV.). Join $O_2 W_5$ and $O_3 W_5$.

Then the trapezium $o_1 O_3 W_5 O_2$ is a face of the *twelve-faced trapezohedron* derived from the *three-faced octahedron* whose face is $o_1 O_2 O_3$.

Twelve of these trapeziums form a net for the twelve-faced trapezohedron which can be inscribed in the cube whose faces are equal to the square $O_1 O_4 O_8 O_5$ (fig. 27, Plate IV.).

The faces of the three-faced octahedron are shaded on those of the twelve-faced trapezohedron (fig. 18, Plate III.).

The twelve-faced trapezohedron derived from the three-faced octahedron 1 1 2, whose symbols are $2\,O$ Naumann, 1 2 2 Miller, and $a^{\frac{1}{2}}$ Brooke; whose symbols are $\frac{1}{2}(1\ 2\ 2)$; $\frac{2O}{2}$ Naumann, κ 1 2 2 Miller, $\frac{1}{2}(a^{\frac{1}{2}})$ Brooke, occurs parallel to faces of crystals of blende, diamond, and pharmacosiderite.

One derived from the three-faced octahedron 1 1 $\frac{3}{2}$, $\frac{3}{2}O$ Naumann, 2 3 3 Miller, and $a^{\frac{2}{3}}$ Brooke, whose symbols are respectively $\frac{1}{2}(1\ 1\ \frac{3}{2})$; $\frac{\frac{3}{2}O}{2}$; κ 2 3 3; and $\frac{1}{2}(a^{\frac{2}{3}})$, occurs parallel to faces of crystals of fahlerz.

136. The *three-faced tetrahedron* is a half-symmetrical form, with inclined faces derived from the *twenty-four-faced trapezohedron*. It is bounded by twelve equal and similar isosceles triangles (figs. 19 and 20, Plate III.).

It is also called the *trigonal dodecahedron, hemi-icositetrahedron, triakis-tetrahedron, pyramidal tetrahedron*, and *kuproid*.

It is formed by producing the three faces of the *twenty-four-faced trapezohedron*, corresponding to each face of the octahedron which are produced to form the tetrahedron, to form a solid by their intersection.

Thus, comparing (figs. 19 and 20, Plate III.) with (fig. 4, Plate I.), the three faces of the twenty-four-faced trapezohedron, meeting respectively in o_1, o_3, o_6, and o_8 (fig. 4), are produced to meet in the points O_2, O_4, O_5, and O_7 (figs. 19 and 20, Plate III.), making by their intersections a *three-faced tetrahedron*, bounded by twelve equal and similar isosceles triangles, $O_4 O_2 o_1$, $O_4 O_5 o_1$, &c.

If we call this the *positive three-faced octahedron*, the *negative* will be formed by the twelve faces of the twenty-four-faced trapezohedron which meet in groups of three in the points o_2, o_4, o_5, and o_7.

To obtain a face of the three-faced tetrahedron geometrically from the twenty-four-faced trapezohedron from which it is derived. Describe the (fig. 31, Plate IV.) as previously constructed, § 61, for determining a face of the twenty-four-faced trapezohedron. Produce C_1A to C_6, O_1D_5 to O_5; make $AC_6 = D_5O_5 = AC_1$. Join C_6O_5, AO_5. Then it will be found that O_1d_5 produced will cut C_6O_5 in O_5.

Let $C_1d_1o_1d_2$ (fig. 39) be the face of the twenty-four-faced trapezohedron derived from (fig. 31, Plate IV.).

Produce o_1d_2 to O_2, and O_1d_1 to O_4, making $o_1d_2O_2$ and $o_1d_1O_4$ equal to $o_1d_5O_5$ (fig. 31). Join O_4O_2; this line will pass through C_1.

Then $O_4O_2o_1$ is a face of the *three-faced octahedron* derived from that of the *twenty-four-faced trapezohedron* whose face is $C_1d_1o_1d_2$.

Twelve of these isosceles triangles form a net for the *three-faced tetrahedron* which can be inscribed in the cube whose faces are equal to the square $O_1O_4O_8O_5$ (fig. 27, Plate IV.).

The faces of the twenty-four-faced trapezohedron are shaded on those of the three-faced tetrahedron (fig. 20, Plate IV.).

The following curious reciprocal relations may be observed between the perfectly symmetrical and half-symmetrical forms of the three-faced octahedron and the twenty-four-faced trapezohedron.

The hemihedral form of the three-faced octahedron is bounded by trapeziums similar to the faces of the twenty-four-faced trapezohedron.

The hemihedral form of the twenty-four-faced trapezohedron is bounded by isosceles triangles like the faces of the three-faced cube.

The three-faced octahedron is formed by placing a three-faced pyramid of equal isosceles triangles on each of the equilateral triangular faces of the regular octahedron as bases. The three-faced tetrahedron is formed in like manner by placing a three-faced pyramid of equal isosceles triangles on each of the equilateral triangular faces of the regular tetrahedron.

The following *three-faced tetrahedrons*, having faces of crystals parallel to them, have been observed in nature :—

$\frac{1}{2}(1\ \frac{3}{2}\ \frac{3}{2})$; $\frac{\frac{3}{2}O\frac{3}{2}}{2}$ Naumann, κ 2 3 3 Miller, $a^{\frac{3}{2}}$ Brooke; in tennantite.

$\frac{1}{2}(1\ 2\ 2)$; $\frac{2O2}{2}$ Naumann, κ 1 1 2 Miller, a^2 Brooke; in boracite, eulytine, fahlerz, and tennantite.

$\frac{1}{2}(1\ 3\ 3)$; $\frac{3O3}{2}$; κ 1 1 3; a^3; in blende and fahlerz.

$\frac{1}{2}(1\ 4\ 4)$; $\frac{4O4}{2}$; $\kappa\ 1\ 1\ 4$; a^4; in blende.

$\frac{1}{2}(1\ 5\ 5)$; $\frac{5O5}{2}$; $\kappa\ 1\ 1\ 5$; a^5; in blende.

137. The *six-faced tetrahedron* is a half-symmetrical form with inclined faces derived from the six-faced octahedron. It is bounded by twenty-four equal and similar scalene triangles (figs. 21 and 22, Plate III.).

It is also called the *hemi-hex-octahedron, hexakis-tetrahedron,* and *boracitoid.*

It is formed by producing the six faces of the six-faced octahedron, corresponding to each face of the octahedron which are produced to form the tetrahedron, to form a solid by their intersection. Thus, comparing (figs. 21 and 22, Plate III.) with (fig. 3, Plate I.), the six faces of the six-faced octahedron, meeting respectively in o_1, o_3, o_6, and o_8 (fig. 3, Plate I.), are produced to meet in the points W_2, W_4, W_5, and W_7 (figs. 21 and 22, Plate III.), making by their intersections a six-faced tetrahedron, bounded by 24 equal and similar scalene triangles, $o_1C_1W_2$, $o_1C_3W_2$, &c.

If we call this the *positive* six-faced tetrahedron, the *negative* will be formed by the twenty-four faces of the six-faced octahedron which meet in groups of six in the points o_2, o_4, o_5, and o_7 (fig. 3, Plate I.). To obtain geometrically a face of the six-faced tetrahedron from the six-faced octahedron from which it is derived, describe the (fig. 35, Plate IV.), as previously constructed, § 68, for determining a face of the six-faced octahedron. Produce C_1A to C_6, O_1D_5 to O_5; make $AC_6=D_5O_5=C_1A$. Join C_6O_5 and AO_5. Produce No_1d_5 to meet AO_5 in W_5, and join C_6W_5.

Then (fig. 36, Plate IV.) let $C_1o_1d_2$ be a face of the six-faced octahedron constructed as in § 69.

Produce o_1d_2 to W_2 and make $o_1d_2W_2=o_1d_5W_5$, fig. 35.

Join C_1W_2. Then the scalene triangle $o_1W_2C_1$ is a face of the six-faced tetrahedron derived from the six-faced octahedron whose face is $C_1o_1d_2$. Twenty-four such scalene triangles form a net for the six-faced tetrahedron which can be inscribed in the cube whose faces are equal to the square $O_1O_4O_8O_5$ (fig. 27, Plate IV.). The faces of the six-faced octahedron are shaded on those of the six-faced tetrahedron (fig. 22, Plate III.).

The following six-faced tetrahedrons, having faces of crystals parallel to them, have been observed in nature:

$\frac{1}{2}(1\ \frac{3}{2}\ 3)$; $\frac{3O\frac{3}{2}}{2}$ Naumann; $\kappa\ 3\ 2\ 1$ Miller; $\frac{1}{2}(b^1\ b^{\frac{1}{2}}\ b^{\frac{1}{3}})$ Brooke; in crystals of the diamond,

$\frac{1}{2}(1\,\frac{5}{3}\,5)$ Naumann; $\frac{5\,O\,\frac{5}{3}}{2}$; κ 5 3 1 Miller; $\frac{1}{2}(b\ {}^1b^{\frac{1}{3}}\ b^{\frac{1}{5}})$ Brooke; in crystals of boracite.

By the construction fig. 35, the ratio $\frac{AW_5}{AO_5}$ may be readily determined by plain trigonometry, just as the ratio $\frac{Ao_1}{AO_1}$ was in § 73.

It can also be readily determined by geometry of three dimensions. For (fig. 22, Plate III.) W_2 is a point in each of the three planes $C_1o_1d_2$, $C_3o_1d_2$, $C_1o_3d_3$.

Now the equation to the plane $C_1o_1d_2$ referred to rectangular co-ordinates, AC_1, AC_2, AC_3, is

$$\frac{x}{m}+\frac{y}{n}+\frac{z}{1}=1 \quad \text{(A)}$$

To the plane $C_3o_1d_2$ is $\frac{x}{1}+\frac{y}{n}+\frac{z}{m}=1 \quad$ (B)

To the plane $C_1o_3d_3$ is $-\frac{x}{n}-\frac{y}{m}+\frac{z}{1}=1 \quad$ (C). (*See* fig. 31*, and fig. 32*, Plate IV.*)

And since x, y, z will be the same for the point W_2 where these planes meet,

$$\text{(A)}-\text{(C)} \quad x\left(\frac{1}{m}+\frac{1}{n}\right)+y\left(\frac{1}{n}+\frac{1}{m}\right)=0.$$

Therefore $x=\ -y$.

$$\text{Also (A}-\text{B)} \quad x\left(\frac{1}{m}-1\right)+z\left(1-\frac{1}{m}\right)=0.$$

And $x=z$.

$$x=-y=z=\frac{1}{1+\frac{1}{m}-\frac{1}{n}}$$

$$\text{But } AW_2^{\,2}=x^2+y^2+z^2=\frac{3}{\left(1+\frac{1}{m}-\frac{1}{n}\right)^2}$$

$$\text{And } AW_2=\frac{\sqrt{3}}{1+\frac{1}{m}-\frac{1}{n}}=\frac{AO_1}{1+\frac{1}{m}-\frac{1}{n}}$$

Again, let ω be the angle which the normals of the faces $C_1o_1d_2$, $C_1o_3d_3$ make with each other, or $180°-\omega$ be the angle of inclination of the two faces of the six-faced tetrahedron (fig. 21, Plate III.), over the edge C_1W_2.

Then since $m\,n\,1$ is the symbol of $C_1o_1d_2$,
and $-n-m\,1$ that of $C_1o_3d_3$,

$$\cos\,\omega = \frac{1-\dfrac{2}{mn}}{1+\dfrac{1}{m^2}+\dfrac{1}{n^2}}\quad (\textit{See}\ \S\ 107.)$$

Or by § 110,

$$\cos\,\omega = -\cos p_2 \cos p_3 - \cos p_2 \cos p_3 + \cos p_1 \cos p_1$$
$$= \cos^2 p_1 - 2\cos p_2 \cos p_3.$$

Which may be computed at once by Byrne's dual logarithms, or thus adapted for ordinary logarithmic computation.

$$\cos\,\omega = \cos^2 p_1 \left\{ 1 - \frac{2\cos p_2 \cos p_3}{\cos^2 p_1} \right\}$$

Let $\tan a = \dfrac{2\cos p_2 \cos p_3}{\cos^2 p_1} = \dfrac{\cos p_2 \cos p_3}{\cos 60 \cos^2 p_1}$

Then $\cos\,\omega = \cos^2 p_1\,(1-\tan a) = \dfrac{\cos^2 p_1 \cos\,(a+45)^\circ}{\cos a \sin 45^\circ}$

138. *Limits of the Form of the Six-faced Tetrahedron.*

As m and n approach in magnitude to unity, the six-faced tetrahedron approximates to the tetrahedron. When $m=n=1$, the six-faced tetrahedron becomes the tetrahedron, the points W_1, W_2, W_5, and W_7 (fig. 21, Plate III.) coincide with the points O_1, O_2, O_5, and O_7 (fig. 15). C_1W_4 and C_1W_2 become the straight line O_2O_4, &c., and the six faces round each point o_1, o_3, o_6, and o_8 lie in the same plane.

As m and n increase in magnitude greater than unity, and also in equality to each other, the six-faced octahedron approximates to the cube. When m and n are both infinitely great, it coincides with it. In this case each of the four faces which meet in the six points C_1, C_2, C_3, &c., C_6, lie in the same plane. As m approaches to unity, while n increases in magnitude, the six-faced tetrahedron approximates to the rhombic dodecahedron. When $m=1$ and $n=\infty$ it becomes the rhombic dodecahedron, and the two faces which lie on each side of the twelve lines W_2o_1, W_4o_1, W_5o_1, &c., lie in the same plane, and the Co and CW become equal.

When m equals unity, while n remains finite, the six-faced tetrahedron becomes the twelve-faced trapezohedron, and the faces on each side of the twelve edges W_2O_1 lie in the same plane, but the edges Co and CW are not equal.

When m and n are equal to each other, both finite and greater than unity, the six-faced tetrahedron becomes the three-faced tetrahedron, and the faces on each side the twelve lines C_1o_1, C_3o_1, C_2o_1, &c., lie in the same plane. W coincides with O and WCW becomes a straight line. When m remains finite, and n becomes infinite, the six-faced octahedron becomes the four-faced cube, and its scalene triangles become isosceles.

From the above it follows that the cube, rhombic dodecahedron, and four-faced cube, which have no hemihedral forms with inclined faces, are limiting forms of the six-faced tetrahedron.

Also that all the formulæ of the tetrahedron, three-faced tetrahedron, and twelve-faced trapezohedron may be derived from those of the six-faced octahedron by giving the proper values to m and n.

139. Table showing the symbols and formulæ of the half-symmetrical forms which are not included in the table § 131, for the holohedral forms. The letters refer to holohedral forms, § 131.

SIX-FACED OCTAHEDRON.

	Naumann.	Miller.	Brooke.	Ratio $\frac{AW}{AO}$	Angle ω.
$H\,\frac{1}{2}(1\,\frac{3}{2}\,3)$	$\frac{3\,O\,\frac{3}{2}}{2}$	κ 3 2 1	$\frac{1}{2}(b^1 b^{\frac{1}{2}} b^{\frac{1}{3}})$	$\frac{3}{4}$	69° 5′
$L\,\frac{1}{2}(1\,\frac{5}{3}\,5)$	$\frac{5\,O\,\frac{5}{3}}{2}$	κ 5 3 1	$\frac{1}{2}(b^1 b^{\frac{1}{3}} b^{\frac{1}{5}})$	$\frac{5}{7}$	57 7
THREE-FACED TETRAHEDRON.					
$e\,\frac{1}{2}(1\,\frac{3}{2}\,\frac{3}{2})$	$\frac{\frac{3}{2}\,O\,\frac{3}{2}}{2}$	κ 2 2 3	$\frac{1}{2}(a^{\frac{3}{2}})$	1	86° 38′
$f\,\frac{1}{2}(1\,2\,2)$	$\frac{2\,O\,2}{2}$	κ 1 1 2	$\frac{1}{2}(a^2)$	1	70 32
$k\,\frac{1}{2}(1\,3\,3)$	$\frac{3\,O\,3}{2}$	κ 1 1 3	$\frac{1}{2}(a^3)$	1	50 29
$l\,\frac{1}{2}(1\,4\,4)$	$\frac{4\,O\,4}{2}$	κ 1 1 4	$\frac{1}{2}(a^4)$	1	38 57
$m\,\frac{1}{2}(1\,5\,5)$	$\frac{5\,O\,5}{2}$	κ 1 1 5	$\frac{1}{2}(a^5)$	1	31 35
TWELVE-FACED TRAPEZOHEDRON.					
$\mathrm{f}\,\frac{1}{2}(1\,1\,\frac{3}{2})$	$\frac{\frac{3}{2}\,O}{2}$	κ 2 3 3	$\frac{1}{2}(a^{\frac{2}{3}})$	$\frac{3}{4}$	97° 51′
$\mathrm{h}\,\frac{1}{2}(1\,1\,2)$	$\frac{2\,O}{2}$	κ 1 1 2	$\frac{1}{2}(a^{\frac{1}{2}})$	$\frac{2}{3}$	90 0
TETRAHEDRON.					
$\mathrm{o}\,\frac{1}{2}(1\,1\,1)$	$\frac{O}{2}$	κ 1 1 1	$\frac{1}{2}(a^1)$	1	109° 28′

140. The *pentagonal dodecahedron* is a half-symmetrical form with parallel faces derived from the four-faced cube. It

is bounded by twelve equal and similar pentagons. These pentagons are, except in one species of the pentagonal dodecahedron, irregular (figs. 23 and 24, Plate III.); four edges or sides of the pentagon being equal, and the fifth unequal. When the five edges are equal, the pentagonal dodecahedron is called the *regular pentagonal dodecahedron,* and is one of the five Platonic bodies.

It is also called the *hemi-hexa-tetrahedron* and *pyritoid.*

It is formed from the *four-faced cube* by taking three out of the six faces (fig. 2, Plate I.) which meet in the points o_1, o_2, &c., o_8; taking the faces alternately and producing them to form by their intersections a solid by twelve pentagonal faces.

Thus the faces $C_1o_1o_4$, $C_1o_2o_3$, $C_2o_1o_5$, $C_2o_4o_8$, $C_3o_1o_2$, $C_3o_5o_6$, $C_4o_2o_6$, $C_4o_3o_7$, $C_5o_4o_3$, $C_5o_7o_8$, $C_6o_5o_8$, and $C_6o_6o_7$ are produced to form the *positive* pentagonal dodecahedron; the twelve remaining faces to form the *negative* pentagonal dodecahedron. The faces so produced meet in twenty-four equal edges $o_1\delta_1$, $o_1\delta_2$, &c. (figs. 23 and 24, Plate III.); and six other edges, but unequal to the former $\delta_1\delta_9$, $\delta_2\delta_4$, &c.

To obtain a face of the pentagonal dodecahedron geometrically from that of the four-faced cube from which it is derived (fig. 37, Plate IV.), being described as in § 53. Produce C_1d_1 to meet D_1C_2 in δ_1.

Describe $C_1o_1o_4$ as in § 54, a face of the four-faced cube (fig. 34, Plate IV.). Bisect o_1o_4 in d_1. Produce C_1d_1 to δ_1, making $C_1d_1\delta_1 = C_1d_1\delta_1$ (fig. 37). Join $o_1\delta_1$ and $o_4\delta_1$. Through C_1 draw $\delta_4C_1\delta_2$ parallel to o_1o_4.

Then (fig. 34) take $C_1\delta_2$ and $C_1\delta_4$ each equal $C_2\delta_1$ (fig. 37). Join $o_4\delta_4$ and $o_1\delta_2$.

Then $\delta_4\delta_2o_1\delta_1o_4$ is a face of the pentagonal dodecahedron derived from the four-faced cube whose face is $C_1o_4o_1$.

Twelve such pentagonal faces form a net for the pentagonal dodecahedron which can be inscribed in the cube whose faces are equal to the square $O_1O_4O_8O_5$ (fig. 27, Plate IV.).

The faces of the four-faced cube are shaded on those of the pentagonal dodecahedron (fig. 24, Plate IV.).

The following pentagonal dodecahedrons, having faces of crystals parallel to them, have been observed in nature:—

$\frac{1}{2}[1\ \frac{5}{4}\ \infty]$; $\dfrac{\infty O \frac{5}{4}}{2}$ Naumann; π 5 4 0 Miller; $\frac{1}{2}\ b^{\frac{5}{4}}$ Brooke, in pyrite.

$\frac{1}{2}[1\ \frac{4}{3}\ \infty]$; $\dfrac{\infty O \frac{4}{3}}{2}$; π 4 3 0; $\frac{1}{2}\ b^{\frac{4}{3}}$, in pyrite.

$\frac{1}{2}[1\ \frac{3}{2}\ \infty]$; $\dfrac{\infty O \frac{3}{2}}{2}$; π 3 2 0; $\frac{1}{2}\ b^{\frac{3}{2}}$, in pyrite.

$\frac{1}{2}[1\ 2\ \infty]$; $\frac{\infty O 2}{2}$; $\pi\ 2\ 1\ 0$; $\frac{1}{2} b^2$, in cobaltine, cubane, fahlerz, gersdorfitte, and pyrite.

$\frac{1}{2}[1\ 3\ \infty]$; $\frac{\infty O 3}{2}$; $\pi\ 3\ 1\ 0$; $\frac{1}{2} b^3$, in hauerite, pyrite, and sal ammoniac.

$\frac{1}{2}[1\ 4\ \infty]$; $\frac{\infty O 4}{2}$; $\pi\ 4\ 1\ 0$; $\frac{1}{2} b^4$, in cobaltine and fahlerz.

141. *Platonic bodies.*—There are five solid bodies described by the ancient geometers as regular solids. From their mathematical properties having been investigated by Plato and his followers, they are called the *Platonic bodies.* They have all their faces, edges, and angles, whether plane or solid, equal for each body.

They are the *tetrahedron,* bounded by four equal faces, each being an equilateral triangle ; the *cube,* bounded by six equal squares ; the *octahedron,* bounded by eight equal faces, each being an equilateral triangle; the *pentagonal dodecahedron,* bounded by twelve equal and equilateral pentagons ; and the *icosahedron,* by twenty equal faces, each being an equilateral triangle.

The first three, described by Plato himself, have been observed in natural crystals. The last two, described after his death, have not been observed in nature.

The *regular pentagonal dodecahedron* is that particular case of the pentagonal dodecahedron, where the unequal edge, such as $\delta_2\delta_4$ (fig. 23, Plate III.), is equal to the other four $\delta_2 o_1$, $o_1\delta_1$, $\delta_1 o_4$, and $o_4\delta_4$.

In this case $m = \cot \lambda_3 = \frac{1+\sqrt{5}}{2} = 1{\cdot}618034$,

but $\cot 31^\circ\ 43' = 1{\cdot}618085$.

Hence $\lambda_3 = 31^\circ\ 43'$ true to minutes.

The value of m is generally determined by continued fractions.

Thus $m = \frac{34}{21} = 1{\cdot}619046$ and $\cot 31^\circ\ 42' = 1{\cdot}61914$
$m = \frac{13}{8} = 1{\cdot}625$ $\quad \cot 31^\circ\ 36' = 1{\cdot}62548$
$m = \frac{8}{5} = 1{\cdot}6$ $\quad \cot 32^\circ\ 0' = 1{\cdot}60033$

The *regular icosahedron* is derived from the particular pentagonal dodecahedron in which the edge $\delta_4\delta_2$ = a line joining the points δ_1 and δ_2. In this case

$$m = \cot \lambda_3 = \frac{3+\sqrt{5}}{2} = 2{\cdot}61803 = \cot 20^\circ\ 54',$$

where the ratio for m expressed in its lowest terms is $m = \frac{34}{13}$.

In this particular pentagonal dodecahedron each solid angle at o_1, o_2, &c., o_8, is cut off through the lines $\delta_1\delta_2$, $\delta_2\delta_5$, and $\delta_5\delta_1$, &c., forming a solid bounded by twenty equilateral triangles,—eight being parallel to the faces of the octahedron inscribed in the dodecahedron, and the remaining twelve faces of the pentagonal dodecahedron.

Ozonam, in his Mathematical Recreations, remarks that "The ancient geometricians made a great many geometrical speculations respecting these bodies; and they form almost the whole subject of the last books of Euclid's Elements. They were suggested to the ancients by their believing that these bodies were endowed with mysterious properties, on which the explanation of the most secret phenomena of nature depended."

142. The *irregular twenty-four-faced trapezohedron* is a half-symmetrical form with *parallel* faces derived from the six-faced octahedron. It is called the irregular twenty-four-faced trapezohedron because its trapezoidal faces have only two equal edges, and to distinguish it from the twenty-four-faced trapezohedron, which is a holohedral form and has the four edges of its trapezoidal faces equal in pairs.

It is bounded by twenty-four irregular trapeziums (figs. 25 and 26, Plate II.).

It is also called the *hemi-octakis-hexahedron*, the *trapezoidal icosi-tetrahedron*, the *dyakis dodecahedron*, the *diploid*, and the *diplopyritoid*.

It is formed from the six-faced octahedron by taking three out of the six faces which meet in o_1, o_2, &c., o_8 (fig. 31, Plate I.), and producing them to meet each other and form a solid bounded by twenty-four irregular trapeziums.

Thus (fig. 8, Plate I.) the twenty-four faces $C_1o_1d_1$, $C_2o_1d_5$, $C_3o_1d_2$, $C_2o_4d_8$, $C_1o_4d_1$, $C_3o_4d_4$, &c., are produced to meet in the points δ_1, δ_2, &c., δ_{12} (fig. 25, Plate III.), to form the *positive* irregular twenty-four-faced trapezohedron.

The remaining twenty-four-faces if produced will form the *negative* trapezohedron.

To obtain a face of the irregular twenty-four-faced trapezohedron geometrically from that of the six-faced octahedron from which it is derived.—Describe (fig. 35, Plate IV.), as previously constructed for finding a face of the six-faced octahedron, § 68 and § 137. Join C_2N cutting C_1d_1 produced in δ_1. Let $C_2o_1d_5$ (fig. 38, Plate IV.) be a face of the six-faced octahedron. Produce C_2d_5 to δ_5, and make $C_2d_5\delta_5$, fig. 38, $=C_1d_1\delta_1$ (fig. 35). Join $o_1\delta_5$, on base C_2o_1, describe the triangle $C_2\delta_1o_1$, having $C_2\delta_1=C_2\delta_1$ fig. 35, and $o_1\delta_1=o_1\delta_5$ fig. 38.

$O_1\delta_5C_2\delta_1$ will be a face of the irregular twenty-four-faced trapezohedron, and twenty-four such faces will form a net for the same, which can be inscribed in a cube whose faces are equal to the square $O_1O_5O_8O_4$ (fig. 27, Plate IV.).

The faces of the six-faced octahedron are shaded on those of the irregular twenty-four-faced trapezohedron in (fig. 26, Plate III.).

The following *irregular twenty-four-faced trapezohedrons*, having faces of crystals parallel to them, have been observed in nature.

$\frac{1}{2}[1\ \frac{5}{4}\ \frac{5}{3}]$; $\frac{\frac{5}{3}O\frac{5}{4}}{2}$ Naumann; π 5 4 3 Miller; $b^{\frac{1}{5}}\ b^{\frac{1}{4}}\ b^{\frac{1}{3}}$ Brooke, in crystals of pyrite.

$\frac{1}{2}[1\ \frac{4}{3}\ 2]$; $\frac{2O\frac{4}{3}}{2}$; π 4 3 2; $b^{\frac{1}{4}}\ b^{\frac{1}{3}}\ b^{\frac{1}{2}}$, in linnéite.

$\frac{1}{2}[1\ \frac{15}{11}\ \frac{15}{7}]$; $\frac{\frac{15}{7}O\frac{15}{11}}{2}$; π 15 11 7; $b^{\frac{1}{15}}\ b^{\frac{1}{11}}\ b^{\frac{1}{7}}$, in linnéite.

$\frac{1}{2}[1\ \frac{3}{2}\ 3]$; $\frac{3O\frac{3}{2}}{2}$; π 3 2 1; $b^{\frac{1}{3}}\ b^{\frac{1}{2}}\ b^{1}$, in cobaltine, hauerite, and pyrite.

$\frac{1}{2}[1\ \frac{3}{2}\ 5]$; $\frac{5O\frac{5}{3}}{2}$; π 5 3 1; $b^{\frac{1}{5}}\ b^{\frac{1}{3}}\ b^{1}$, in pyrite.

$\frac{1}{2}[1\ \frac{5}{3}\ 10]$; $\frac{10O\frac{5}{3}}{2}$; π 10 6 1; $b^{\frac{1}{10}}\ b^{\frac{1}{6}}\ b^{1}$, in pyrite.

$\frac{1}{2}[1\ 2\ 4]$; $\frac{4O2}{2}$; π 4 2 1; $b^{\frac{1}{4}}\ b^{\frac{1}{2}}\ b^{1}$, in pyrite.

$\frac{1}{2}[1\ 5\ 10]$; $\frac{10O5}{2}$; π 10 5 1; $b^{\frac{1}{10}}\ b^{\frac{1}{5}}\ b^{1}$, in pyrite.

143. Let μ be the supplement of the angle of adjacent faces over the edges, such as $C_1\delta_2$, $C_2\delta_1$, $C_3\delta_5$, &c.

ν that over the edges $o_1\delta_1$, $o_1\delta_5$, $o_1\delta_2$, &c.

Then μ is the inclination of normal of face $C_2o_1d_5$ to that of $C_2o_4d_8$, fig. 26, Plate III., but indices of $C_2o_1d_5$ are $m\ 1\ n$, and of $C_2o_4d_8$ $\bar{m}\ 1\ n$ (fig. 31*, Plate IV.*).

$$\text{Hence } \cos\mu = \frac{-\frac{1}{m^2}+\frac{1}{1}+\frac{1}{n^2}}{\frac{1}{m^2}+\frac{1}{n^2}+1}$$

Also ν is the inclination of normal of face $C_2d_5o_1$ to that of $C_1d_1o_1$ (fig. 26 Plate III.), but indices of $C_2d_5o_1$ are $m\ 1\ n$, and of $C_1d_1o_1$, $n\ m\ 1$ (fig. 31*, Plate IV.*).

$$\text{Hence } \cos\nu = \frac{\frac{1}{mn}+\frac{1}{m}+\frac{1}{n}}{\frac{1}{m^2}+\frac{1}{m^2}+1}$$

Or, expressing μ and ν in terms of the polar distances $C_2o_1d_5 = p_2p_1p_3$ and $C_2o_4d_5 = -p_2p_1p_3$.

And $\cos\mu = \cos^2 p_1 - \cos^2 p_2 + \cos^2 p_3$,

$$C_2d_5o_1 = p_2p_1p_3 \quad C_1d_1o_1 = p_3p_2p_1,$$

$$\cos\nu = \cos p_2 \cos p_3 + \cos p_1p_2 + \cos p_1p_3;$$

formulæ calculable at once by Byrne's dual logarithims, or easily adapted to logarithmic computation by subsidiary angles.

All the formulæ for the *pentagonal dodecahedrons* are immediately derivable from those of the *irregular twenty-four-faced trapezohedron.*

144. *Limits of the Form of the Irregular Twenty-four-faced Trapezohedron.*

As m and n approach in magnitude to unity, the irregular twenty-four-faced trapezohedron approximates to the octahedron; and when m and n both equal unity, it becomes the octahedron. In this case the three planes meeting in the points o_1, o_2, &c., o_8 (fig. 25, Plate III.), lie in the same plane, and the edges, such as $C_1\delta_1$, $C_2\delta_1$, lie in the same line.

As m and n both increase in magnitude and become infinitely great, this form approximates to and becomes the cube. In this case the four planes meeting in C_1, C_2, &c., C_6, become the same plane, and the edges, such as $o_4\delta_1o_1$, $o_1\delta_5o_5$, &c., the same straight line.

As m approaches to unity while n increases in magnitude and becomes infinitely great, the form approaches the rhombic dodecahedron. When m equals unity, while n remains finite, the form becomes the three-faced octahedron. When m and n equal each other and are both finite and greater than unity, the form becomes that of the regular twenty-four-faced trapezohedron. Finally, when m remains finite and greater than unity and n becomes infinite, the form becomes that of the pentagonal dodecahedron.

145. As yet the half-symmetrical forms with parallel faces, the pentagonal dodecahedron and the irregular twenty-four-faced trapezohedron have only been found in combination with those of the full symmetrical forms of the cubical system, and never with those of the half-symmetrical forms with inclined faces.

146. For the pentagonal dodecahedrons the following are the values of the angles μ and ν.

E $\frac{1}{2}$ $[1\ \frac{5}{4}\ \infty]$ $\mu = 77°\ 19'$ $\nu = 60°\ 48'$.
F $\frac{1}{2}$ $[1\ \frac{4}{3}\ \infty]$ $\mu = 73°\ 44'$ $\nu = 61°\ 19'$.
G $\frac{1}{2}$ $[1\ \frac{3}{2}\ \infty]$ $\mu = 67°\ 23'$ $\nu = 62°\ 31'$.
H $\frac{1}{2}$ $[1\ 2\ \infty]$ $\mu = 53°\ \ 8'$ $\nu = 66°\ 25'$.
M $\frac{1}{2}$ $[1\ 3\ \infty]$ $\mu = 36°\ 52'$ $\nu = 72°\ 33'$.
N $\frac{1}{2}$ $[1\ 4\ \infty]$ $\mu = 28°\ \ 4'$ $\nu = 76°\ 23'$.

For the irregular twenty-four-faced trapezohedrons the following are the values of μ and ν.

$$
\begin{array}{lllll}
B & \tfrac{1}{2}[1\ \ \tfrac{5}{4}\ \ \tfrac{5}{3}] & \mu=68^\circ\,54' & \nu=19^\circ\,57'. \\
E & \tfrac{1}{2}[1\ \ \tfrac{4}{3}\ \ 2] & \mu=67^\circ\,16' & \nu=26^\circ\,17'. \\
G & \tfrac{1}{2}[1\ \ \tfrac{15}{11}\ \ \tfrac{15}{7}] & \mu=67^\circ\,13' & \nu=28^\circ\,32'. \\
H & \tfrac{1}{2}[1\ \ \tfrac{3}{2}\ \ 2] & \mu=64^\circ\,37' & \nu=38^\circ\,13'. \\
K & \tfrac{1}{2}[1\ \ \tfrac{8}{5}\ \ 8] & \mu=63^\circ\,37' & \nu=53^\circ\,55'. \\
L & \tfrac{1}{2}[1\ \ \tfrac{3}{2}\ \ 5] & \mu=60^\circ\,56' & \nu=48^\circ\,55'. \\
M & \tfrac{1}{2}[1\ \ \tfrac{5}{3}\ \ 10] & \mu=61^\circ\,41' & \nu=56^\circ\,18'. \\
N & \tfrac{1}{2}[1\ \ 2\ \ 4] & \mu=51^\circ\,45' & \nu=48^\circ\,11'. \\
P & \tfrac{1}{2}[1\ \ 5\ \ 10] & \mu=22^\circ\,46' & \nu=72^\circ\,17'.
\end{array}
$$

147. Some crystals have a tendency to split in directions parallel to a certain form. This is called a cleavage-plane. If they split readily, the cleavage is called a perfect one. Substances which crystallize in the cubical system have only been observed to split or cleave parallel to the planes of the cube, octahedron, and rhombic dodecahedron.

Minerals whose crystals cleave parallel to the faces of the cube, those printed in italics *indicating that the cleavage is easy and perfect :—*

Alabandine.	*Galena.*	Pyrite.
Altaite.	*Gersdorffite.*	Pyrochlore.
Analcine.	*Hauerite.*	*Salt.*
Argentite.	Iridium.	*Skutterudite.*
Chromite.	Iron.	Smaltine.
Clausthalite.	Lerbachite.	Spinelle.
Cobaltine.	Linnéite.	Stannine.
Cubane.	*Magnetite.*	Steinmannite.
Embolite.	*Naumannite.*	*Sylvine.*
Franklinite.	*Periclase.*	*Ullmanite.*
Gahnite.	Perowskite.	

Minerals whose crystals cleave parallel to the faces of the octahedron :—

Alum.	*Diamond.*	Grünauite.
Arsenite.	Eisennickelkies.	Magnetite.
Boracite.	Fahlerz.	Sal ammoniac.
Bornite.	*Fluor.*	Senarmontite.
Chromite.	Franklinite.	Smaltine.
Cuprite.	Gahnite.	Spinelle.

Minerals whose crystals cleave parallel to the faces of the rhombic dodecahedron :—

Alabandine.	Garnet.	Smaltine.
Amalgam.	*Hauyne.*	Sodalite.
Argentite.	*Ittnerite.*	Stannine.
Blende.	Leucite.	Tennantite.
Eulytine.	Skutterudite.	

148. In the following table all substances which crystallize on the cubical system are arranged according to their chemical formulæ; the letters c, o, and d, representing that faces parallel to the cube, octahedron, and rhombic dodecahedron, occur on their crystals. The crystals having faces parallel to other forms have been previously enumerated under those forms. The table is principally taken from Rammelsberg's Crystallographic Chemistry.

Chemical Formulæ of Substances crystallizing on the Cubical System.

Ag, Silver (o c d)
Au, Gold (o c d)
Cu, Copper (o c)
Fe, Iron (o c)
Hg, Mercury (o)
Ir, Iridium (o c)
Pb, Lead (o)
Pt, Platinum (c)
P, Phosphorus (o d)
C, Diamond (o c d)
$\dot{Mg}$, Periclase (o c)
$\dot{Ni}$ (o c)
$\dot{Cd}$ (o c d)
$\dot{\cancel{Cu}}$, Cuprite (o c d)
$\dddot{Sb}$, Senarmontite (o)
$\dddot{As}$, Arsenite (o)
$\dot{U}\ \dddot{\cancel{U}}$, Pechuran (o)
$\dddot{Ir} + \dddot{Os}$, Irite (o)
$\dot{Ca} + \ddot{Ti}$, Perowskite (o d)
$\dot{Ca} + 4\dddot{B}$, Bhodozite (o d)
$\dot{Fe} + (\dddot{\cancel{Fe}}\ \dddot{\cancel{Ti}})$, Iserine (o c d)
$\cancel{Cu}'$ and Cu′ Fe′ (o)
Mn′, Alabandine (o c d)
Zn′, Blende (o c d)
Pb′, Galena (o c d)
Pb′ Fe′
Pb′ Sb‴, Steinmannite (o c)
Ag′, Argentite (o c d)
Mn″, Hauerite (o c d)
Fe″, Pyrite (o c d)
Ni′, Grünauite (o c)

Ni As, Rammelsbergite (o c d)
Co As, Smaltine (o c d)
$Co^2\ As^3$, Skutterudite (o c d)
$(Ni\ Co)^m\ As^n$
(Co Fe) As, Safflorite (o c)
Ni″ + Ni (Sb As) Cobaltine (o c)
K Fl
Na Fl (c d)
Ca Fl (o c d) Fluor
K Cl, Sylvine (c o)
Am Cl, Salammoniac (o c d)
Na Cl, Salt (c o d)
Li Cl (c)
Ag Cl, Kerate (c o d)
U Cl (c)
$\cancel{Cu}$ Cl
Co Cl + 8 aq (o c)
K Br (c)
Na Br (c)
Ag Br, Bromite (c o)
K I (c o d)
Am I (c o d)
Na I (c)
Zn I (o)
Pb I (o)
K Cy
Am Cy (o c)
Na Cy
$Ti\ Cy + 3Ti^3\ N$ (c)
Ag Hg, Amalgam (o c d)
Ag^6 Hg, Arquerite (o)
Ag Se, Naumannite (c)
Ag Te, Petzite (c)
Pb Se, Clausthalite (c)
Pb Se and Hg Se, Lerbachite (c)

Pb Te, Altaite (c)

$\dot{Mg}+\dddot{\cancel{Al}}$, Spinelle (o d)

$\dot{Zn}+\dddot{\cancel{Al}}$, Gahnite (o c)

$\dot{Fe}+\dddot{\cancel{Fe}}$, Magnetite (o c d)

$\dot{Fe}+\dddot{\cancel{Cr}}$, Chromite (o)

$\dot{Fe}+\dddot{Mn}$, Franklinite o c d

$\dddot{\cancel{Al}}\ \dddot{S}^3+18\ aq$

$\dddot{\cancel{Cr}}\ \dddot{S}^3+15\ aq$

$\dot{Ba}\ \dddot{N}$ (o c)

$\dot{Sr}\ \dddot{N}$ (o c)

$\dot{Pb}\ \dddot{N}$ (o c)

$\dot{Na}\ \dddot{Cl}$ (c o d)

$\dot{Ni}\ \dddot{Cl}+6\ aq$

$\dot{Co}\ \dddot{Cl}+6\ aq$

$\dot{Cu}\ \dddot{Cl}+6\ aq$ (o)

$\dot{K}\ \dddot{Br}$ (c o d)

$\dot{Na}\ \dddot{Br}$ (c o d)

$\dot{Mg}\ \dddot{Br}+6\ aq$

$\dot{Zn}\ \dddot{Br}+6\ aq$

$\dot{Ni}\ \dddot{Br}+6\ aq$

$\dot{Co}\ \dddot{Br}+6\ aq$ (o c)

$\dot{Am}\ \dddot{I}$ (c)

$\dot{Mg}^3\ Br$

$\dot{Mg}^3\ \ddot{B}^4$, Boracite (c o d)

$\dot{Na}\ \ddot{B}^2+5\ aq$, Borax

$\dot{Na}\ \dot{H}+12\ (\dot{Na}\ \dddot{Sb})+7\ aq$ (o)

$3\ (\cancel{Fe}\ \dddot{As}+4\ aq)+H_3\ \cancel{Fe}$, Pharmacosiderite (o c d)

$\cancel{Cu}'\ \cancel{Fe}'''+2$ Fe, Cubane (c)

$\cancel{Cu}'^3\ \cancel{Fe}'''$, Bornite (c o d)

$Co'\ \cancel{Co}'''$, Linnéite (c o)

$Pb^2\ As'''$, Dufrenoysite (d)

$\dot{R}^4\ (Sb'''\ As''')$, Fahlerz (o c d)

$\dot{R}$=Pb, Fe, Zn, and $\cancel{Cu}^4$

$(Ni\ Co)^3\ S^4$

$Ni\ Sb+Ni'''$, Ullmanite (o c d)

$4(Fe'\ 2\,Ca')+As'''$, Tennantite (o c d)

$Na'^3\ Sb'''+18\ aq$ (o d)

$\cancel{Fe}'\ Ni'$, Eisennickelkies (o)

$(2\,\cancel{Cu}'+Sn'')+(\cancel{Fe}'+Sn'')$, Stannine (c d)

$Ni'+Ni\ As^2$, Gersdorffite (o c)

Am Cl+Mn Cl+aq (c d)

Ca Cl+5 Hg Cl+8 aq (o)

$[2\ (K\ Am)\ Cl+\cancel{Fe}\ Cl^3]+2\ aq$

$(Ni\ Cl+2\,N\ H^3)+aq$ (o c d)

$Am\ Cl+Sn\ Cl^2$ (o c d)

$K\ Cl+Pt\ Cl^2$ (o)

$(Pb\ Cl+\dot{Pb})+(Cu\ Cl+\dot{Cu})+aq$, Percylite (o c d)

2 Ag Br+3 Ag Cl, Embolite o c

$Zn\ Br+N\ H^3$ (o)

$Ca\ Br+N\ H^3$ (o)

$Ni\ I+3\,N\ H^3$ (o)

K Cy+Zn Cy (o)

K Cy+Cd Cy (o)

K Cy+Hg Cy (o)

K Cy+Ag Cy (o)

$\dot{K}\ \dddot{S}+\dddot{Al}\ \dddot{S}^3+24\ aq$, Alum (o c d)

$\dot{Am}\ \dddot{S}+\dddot{\cancel{Al}}\ \dddot{S}^3+24\ aq$

$\dot{K}\ \dddot{S}+\dddot{\cancel{Fe}}\ \dddot{S}^3+24\ aq$

$\dot{Am}\ \dddot{S}+\dddot{Fe}\ \dddot{S}^3+24\ aq$

$\dot{K}\ \dddot{S}+\dddot{\cancel{Mn}}\ \dddot{S}^3+24\ aq$

$\dot{Am}\ \dddot{S}+\dddot{\cancel{Mn}}\ \dddot{S}^3+24\ aq$

$\dot{K}\ \dddot{S}+\dddot{\cancel{Cr}}\ \dddot{S}^3+24\ aq$

$\dot{Am}\ \dddot{S}+\dddot{\cancel{Cr}}\ \dddot{S}^3+24\ aq$

$3\,(\dot{Fe}\ \dot{K})\ \dddot{S}+2\,\dddot{Fe}\ \dddot{S}^3+12\ aq$ (o)

$\dddot{\cancel{Bi}}+\dddot{\cancel{Si}}^3$, Eulytine (o c d)

$\dot{Na}\ \dddot{Si}+\dddot{Al}\ \dddot{Si}^3$, Analcine (c)

$\dot{K}\ \dddot{Si}+\dddot{Al}\ \dddot{Si}^3$, Leucite (d)

$\dot{R}^3\ \dddot{Si}^2+\dddot{R}'\ \dddot{Si}$, Garnet (c d)

Where $\dot{R} = \dot{C}a, \dot{F}e, \dot{M}n,$

and $\dddot{R}' = \cancel{Fe}, \dddot{A}l$

$\dot{C}a^3\ \dddot{S}i^2 + \ddot{\cancel{Cr}}\ \dddot{S}i$, Uwarrowite (o d)

$(\dot{M}n, \dot{F}e)^3\ \dddot{S}i^2 + \cancel{Be}\ \dddot{S}i + Am\ S$, Mn O, Helvin (o)

$Na\ Cl + 3\ \dot{N}a\ \dddot{S}i + 3\ \dddot{a}l\ \dddot{S}i$, Sodaltite, (c d)

$(\dot{N}i\ \overset{\cdot\cdot\cdot}{\ddot{N}} + 2N\ H^3) + aq$ (o c)

$(\dot{N}a\ \ddot{C}a + 8\dot{Z}n\ \ddot{C}) + 8aq$ (o)

$(3\dot{N}a\ \ddot{\cancel{C}} + \cancel{Cr}\ \ddot{\cancel{C}}^3) + 9aq$

$\dot{F}e^3\ \overset{\cdot\cdot\cdot}{\ddot{A}}s + \cancel{Fe}^3\ \overset{\cdot\cdot\cdot}{\ddot{A}}s^2 + 18aq$ (c)

$\dot{N}a\ \dddot{W} + \dddot{W}\ \dddot{W}$ (c)

$\dot{N}a\ \dddot{A}c + 2\ \ddot{\cancel{U}}\ \dddot{A}c$ (o)

$C^{12}\ (H^6\ Cl)\ N$

$C^{12}\ (H^6\ Br)\ N$ (o)

$C^{20}\ H^{16}\ O^2$, Camphor (o)

Substances whose formulæ are undetermined:—

Hauyne, or Lapis Lazuli, a silicate of Alumina, Soda, and Lime (o c d)

Pyrochlore, Titanium ore (o c d)

Tritonite, Silicate of oxides of Cerium and Lanthanium (c)

Voltaite, Hydrous sulphate of iron, &c. (o c d)

*** A discussion* followed, in which C. Brooke, Esq., F.R.S., Professor Morris, the Honorary Secretary, and the Chairman took part; after which—

The Meeting was adjourned.

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* This discussion having been of a very general character, it has not been found necessary to insert it.

Plate I.

Fig 1. 1.∞.∞.

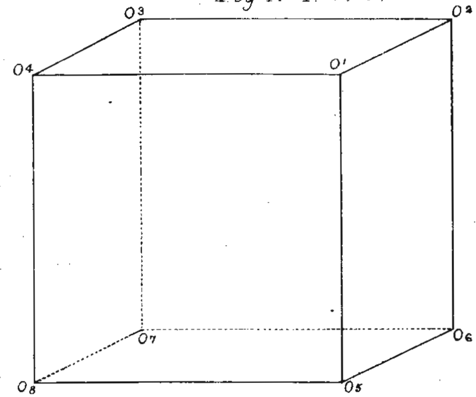

Fig. 2. 1.m.∞.

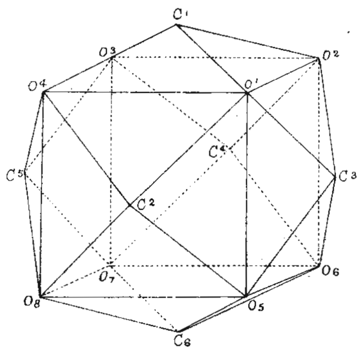

Fig. 3. 1.m.n.

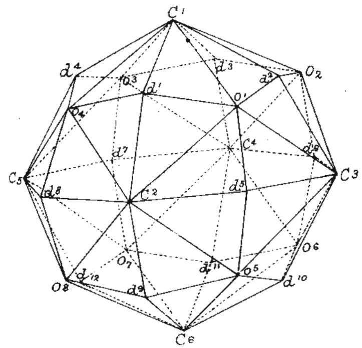

Fig. 4. 1.m.m.

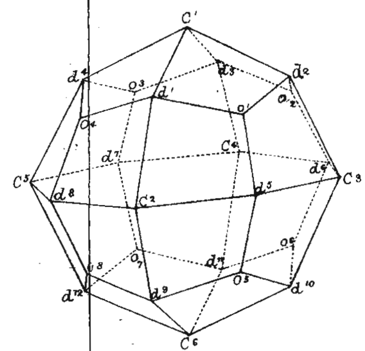

Fig. 5. 1.1.∞.

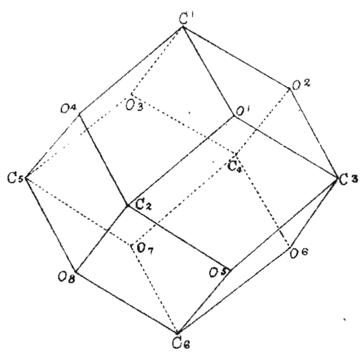

Fig. 6. 1.1.m.

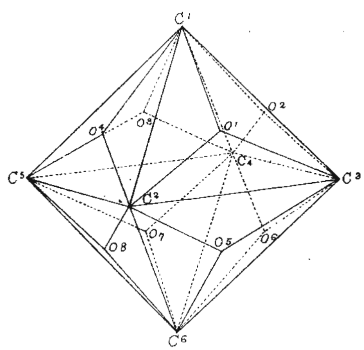

Fig. 7. 1.1.1.

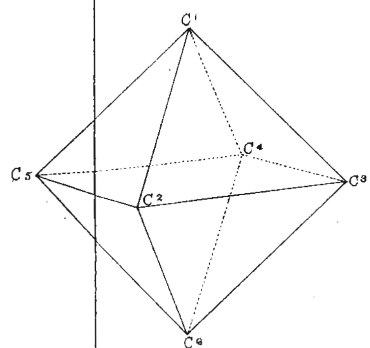

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Plate I.

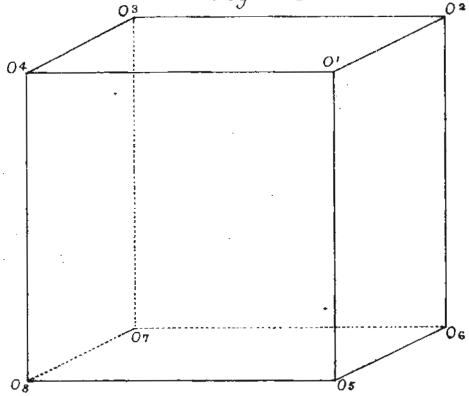

Fig 1. 1.∞.∞.

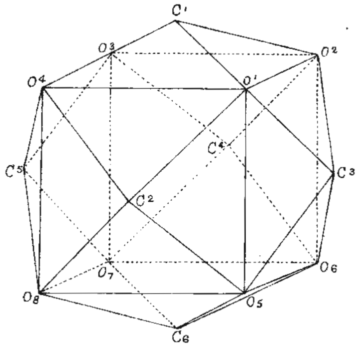

Fig. 2. 1. m. ∞.

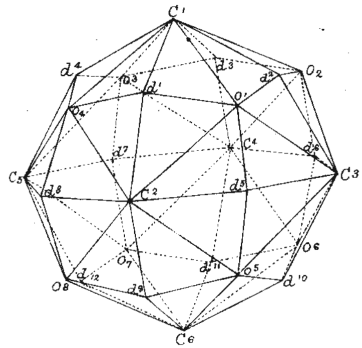

Fig. 3. 1.m.n.

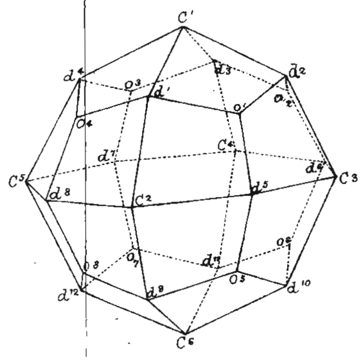

Fig. 4. 1.m.m.

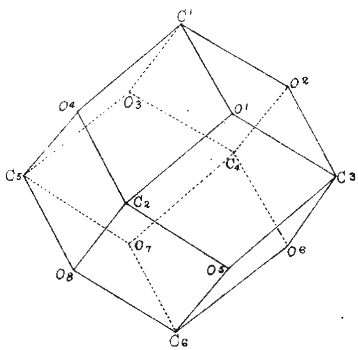

Fig. 5. 1.1.∞.

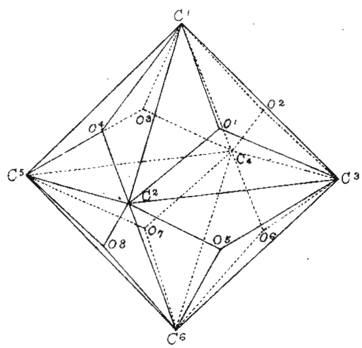

Fig. 6. 1.1. m.

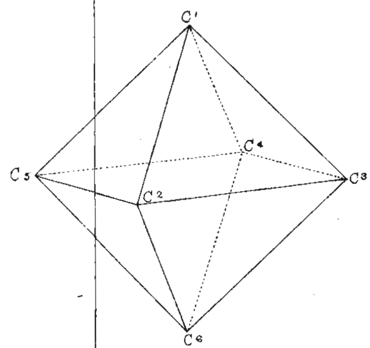

Fig. 7. 1.1.1.

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Plate II.

Fig. 8.

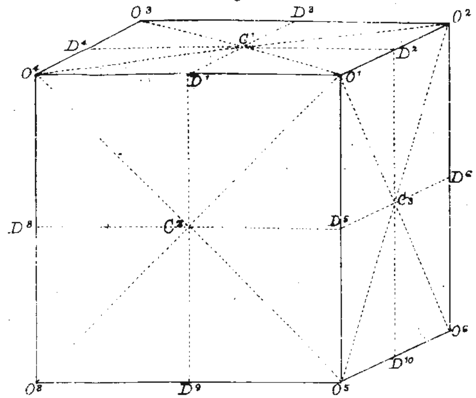

Fig. 9

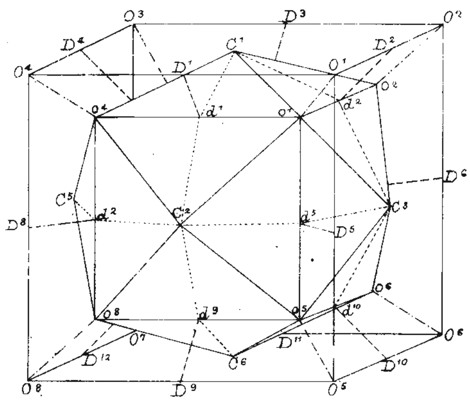

Fig. 10

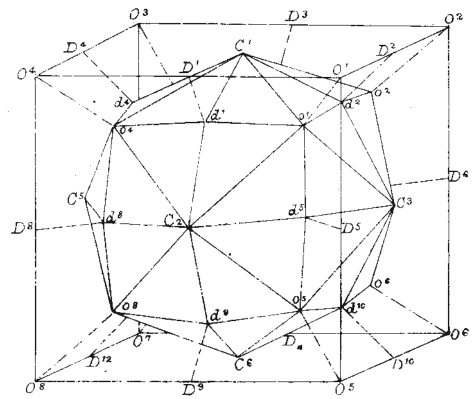

Fig. 11.

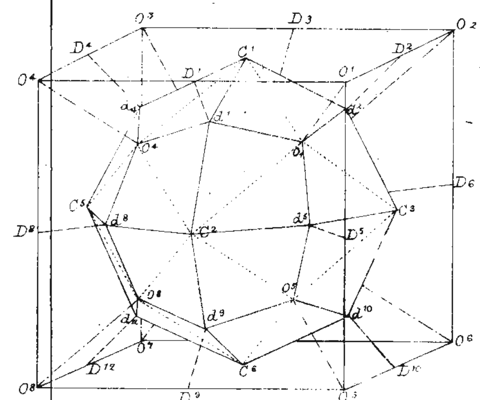

Fig. 12

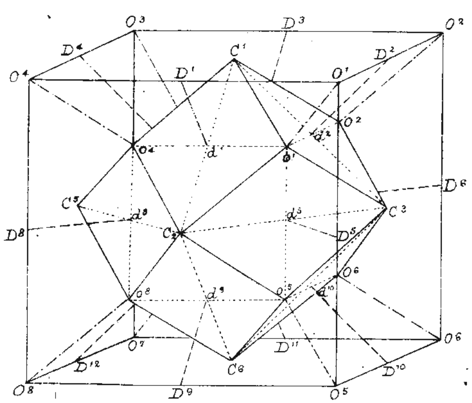

Fig. 13.

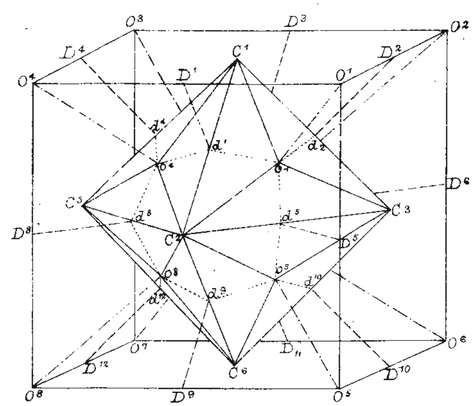

Fig. 14.

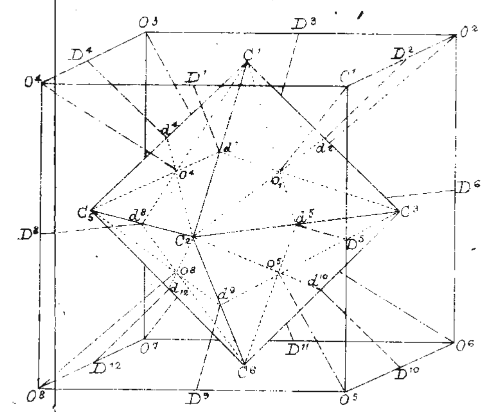

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Plate III

Fig. 15.

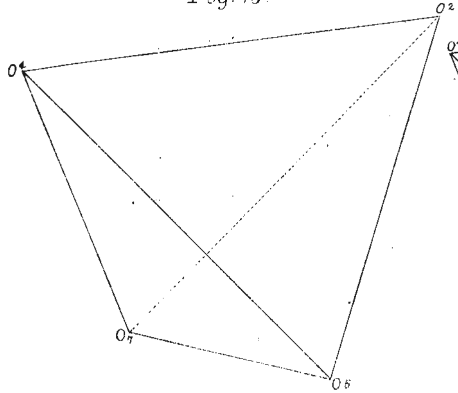


Fig. 16.

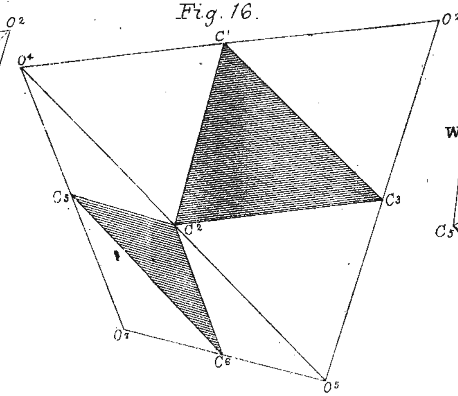


Fig. 17.

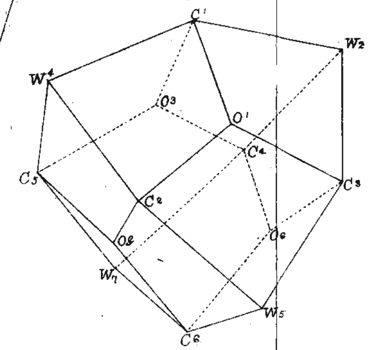
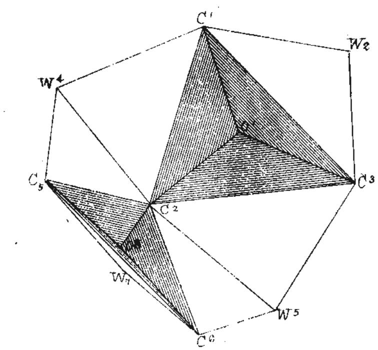


Fig 18.



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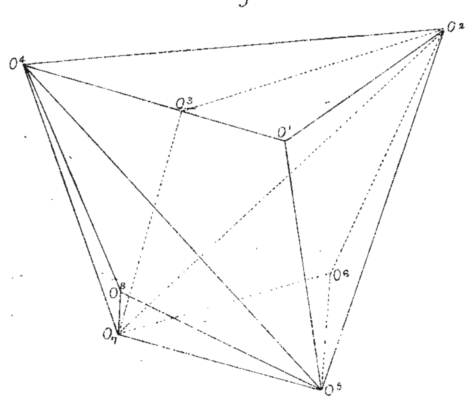

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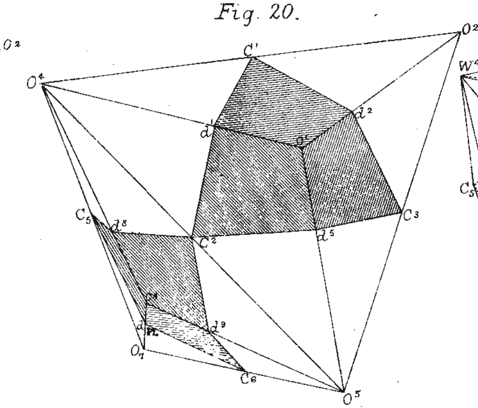


Fig. 21.

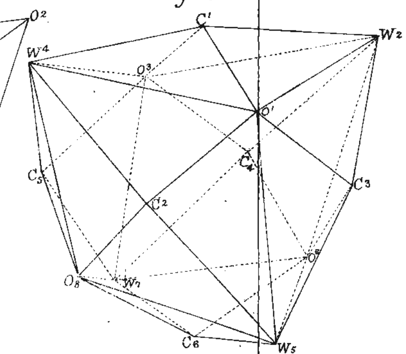


Fig. 22.

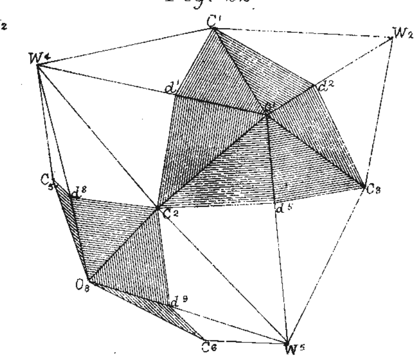
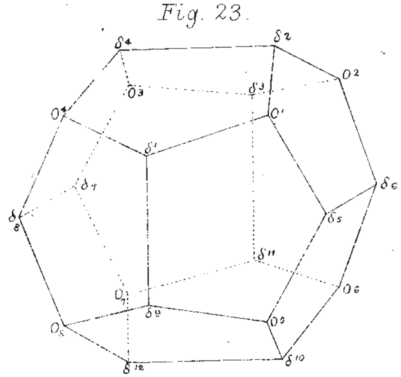


Fig. 23.



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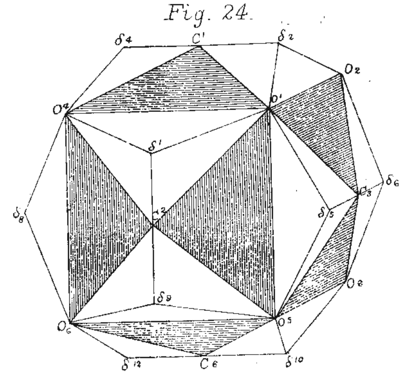

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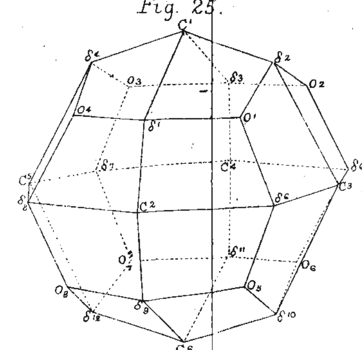

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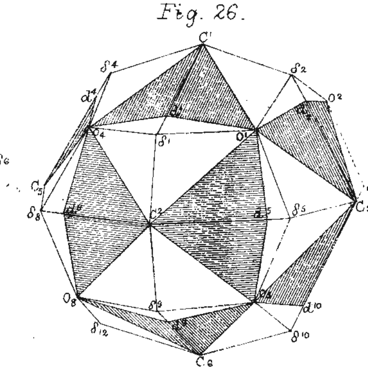

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Plate IV.

Fig. 27.

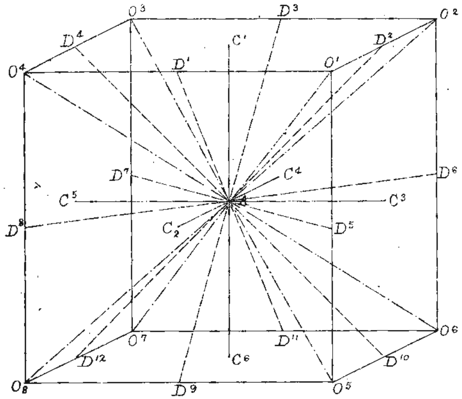
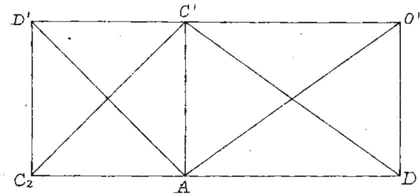


Fig. 28



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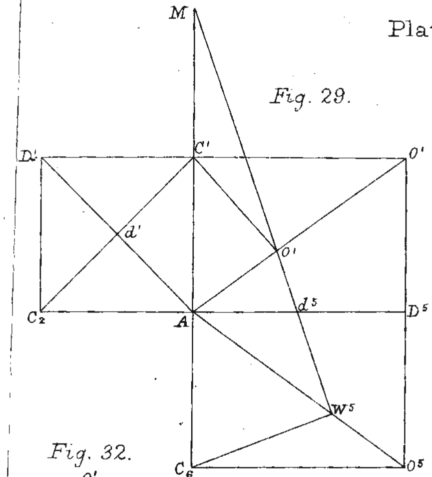

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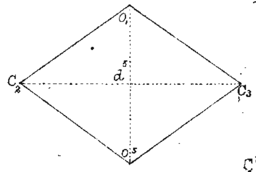

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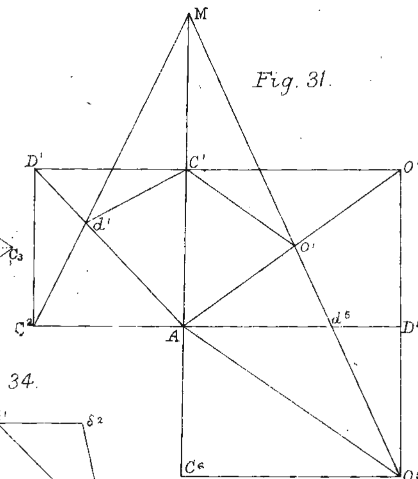

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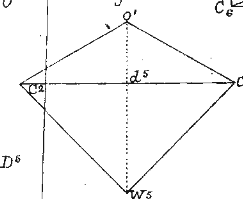
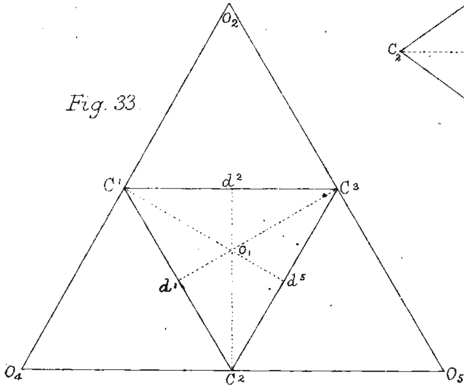


Fig. 33



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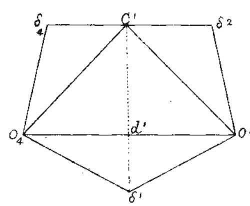

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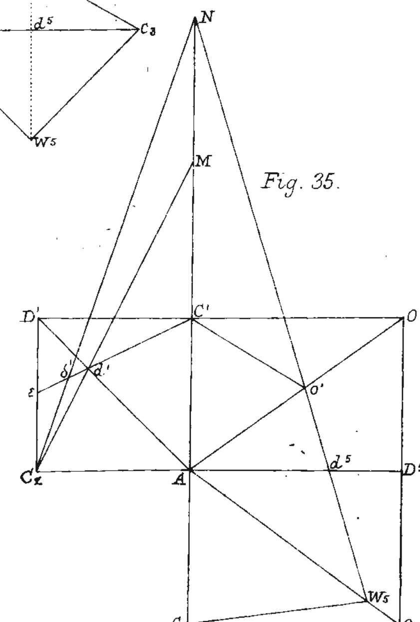

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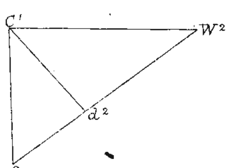

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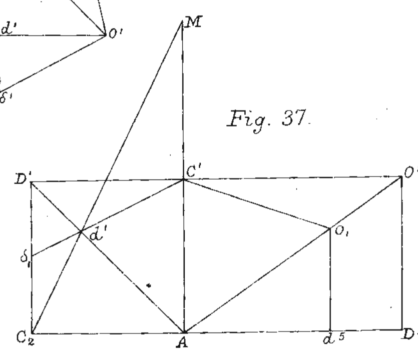

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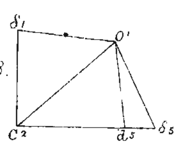

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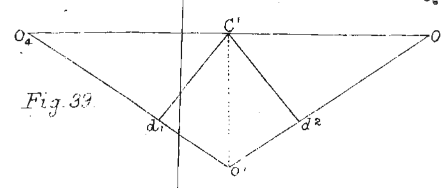

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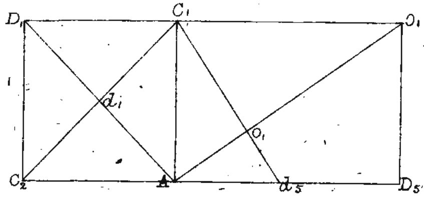
Fig. 27. *



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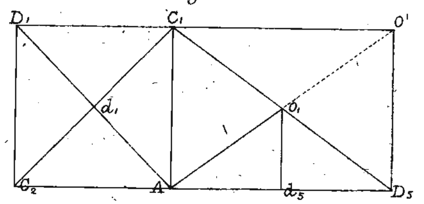

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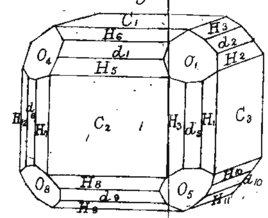

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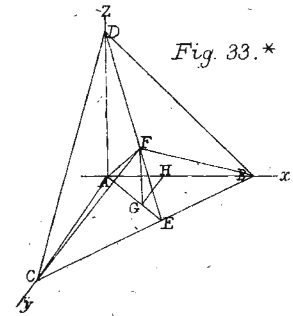

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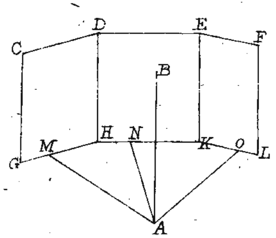

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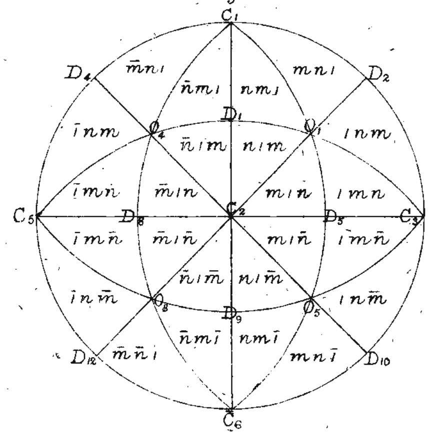

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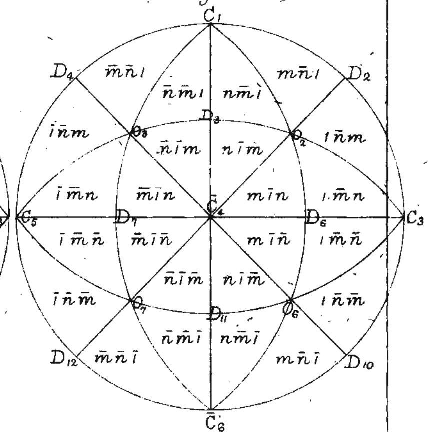

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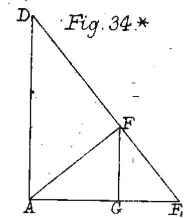

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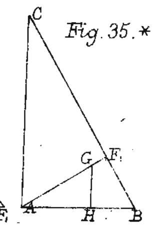

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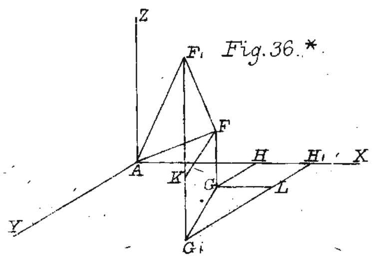

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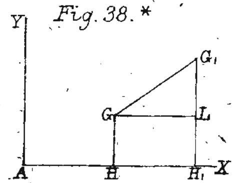

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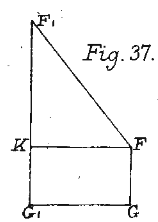

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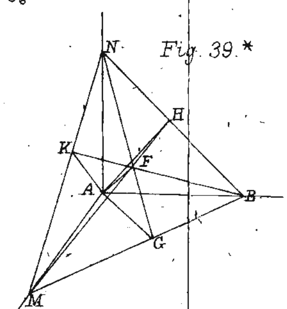

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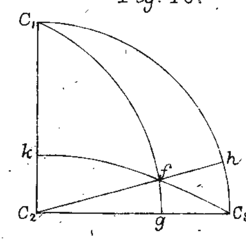

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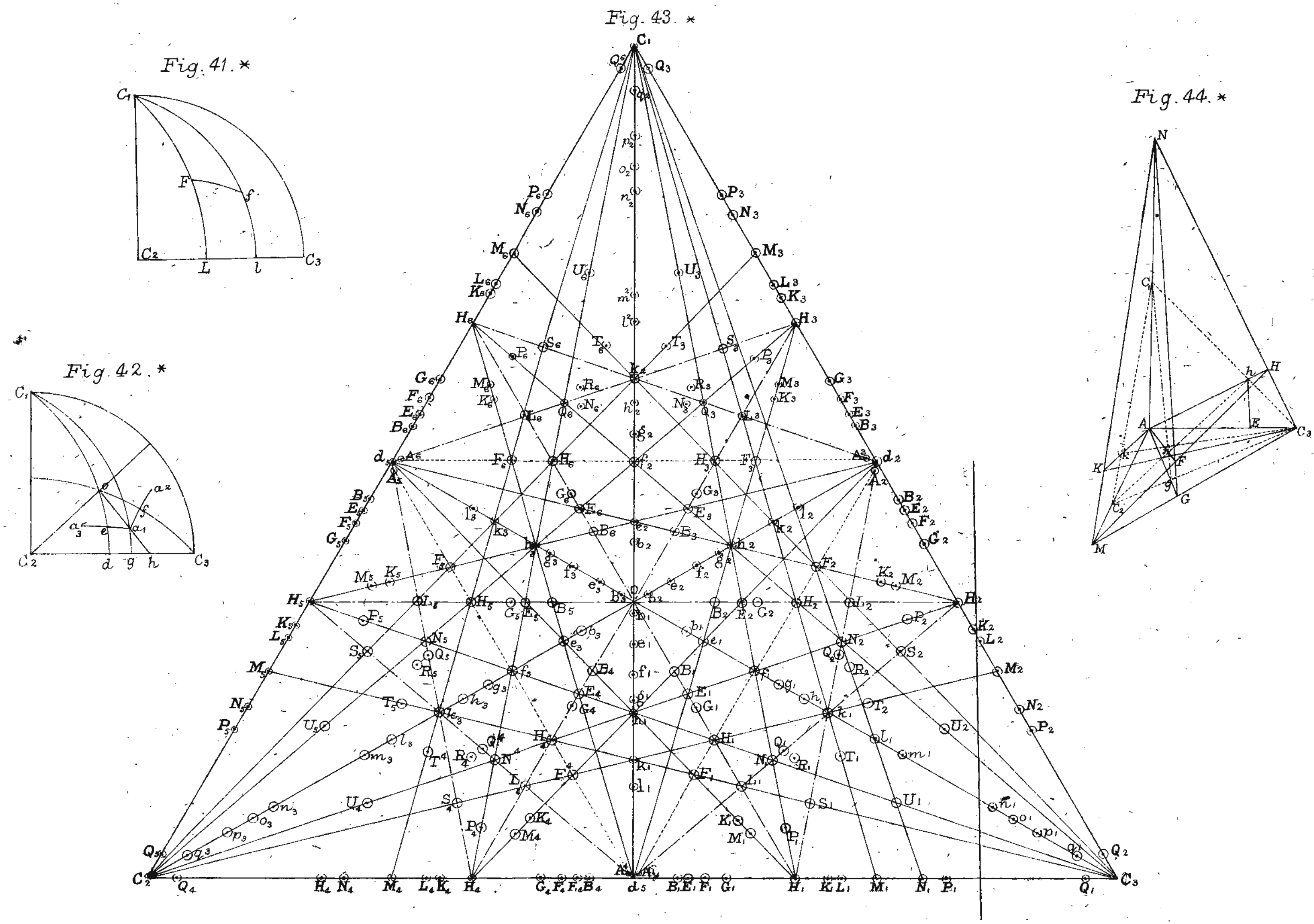

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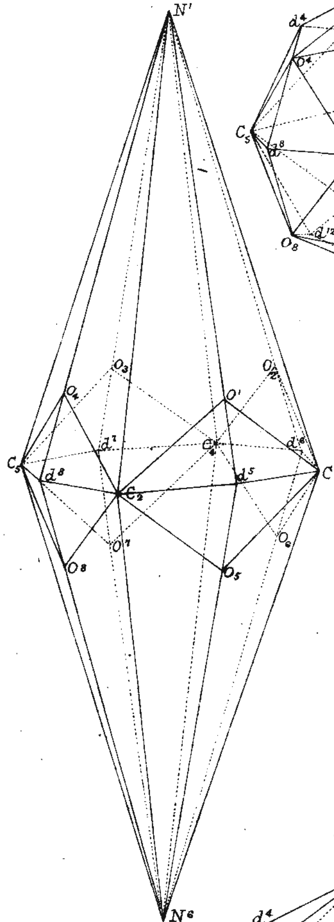
Plate V.

Fig. 41.



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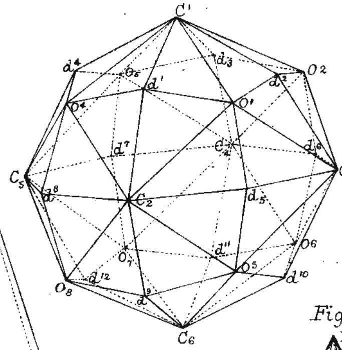

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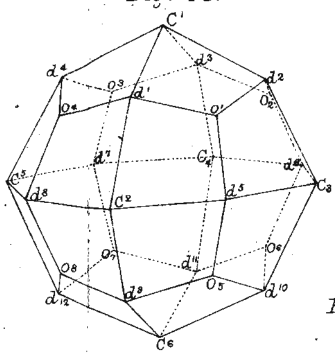
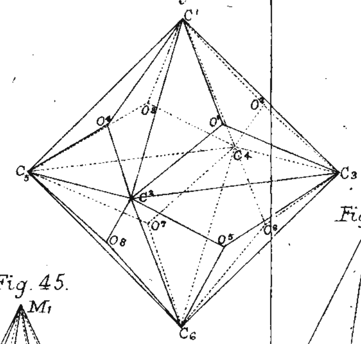


Fig. 47



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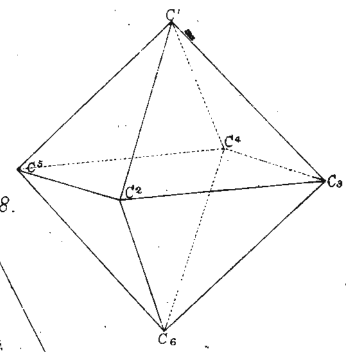

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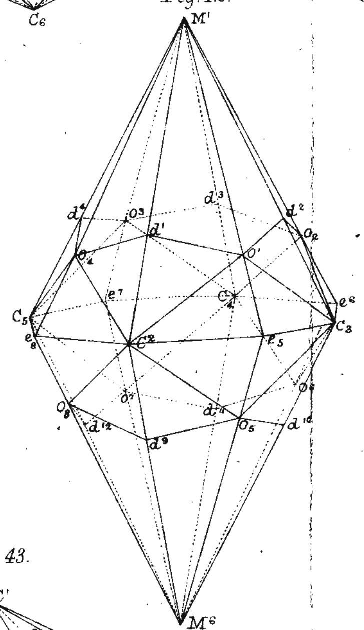

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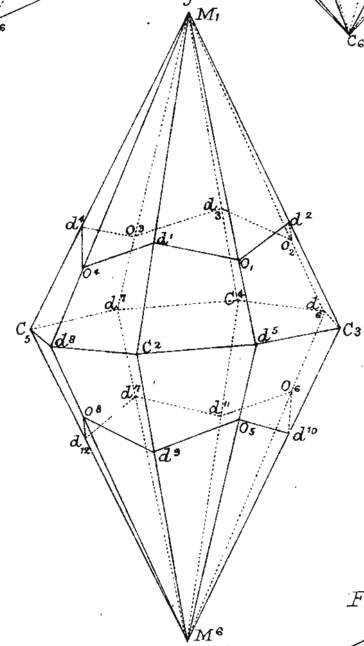

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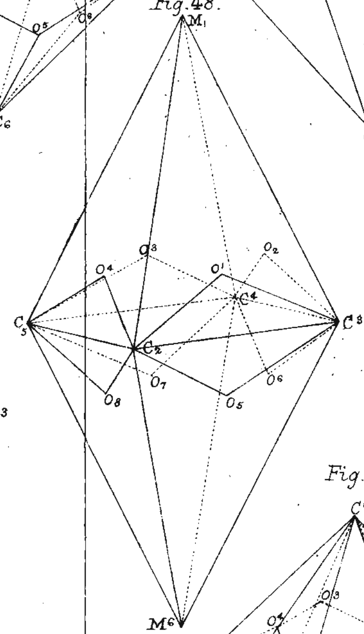

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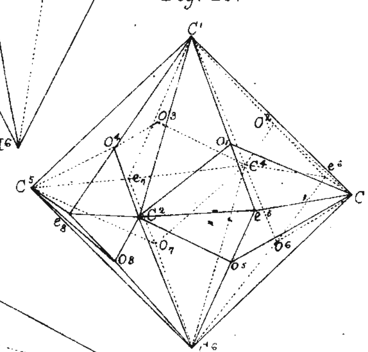

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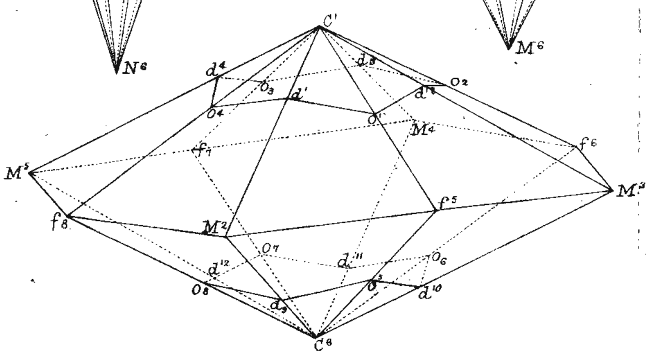

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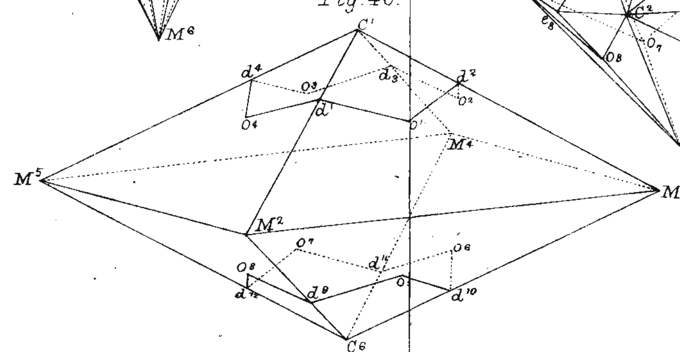

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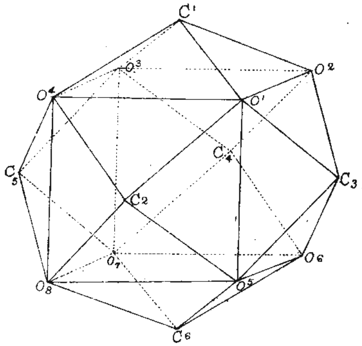
Fig. 51.



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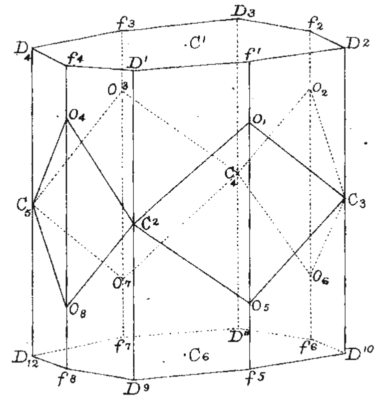

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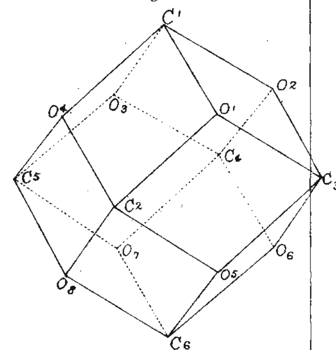

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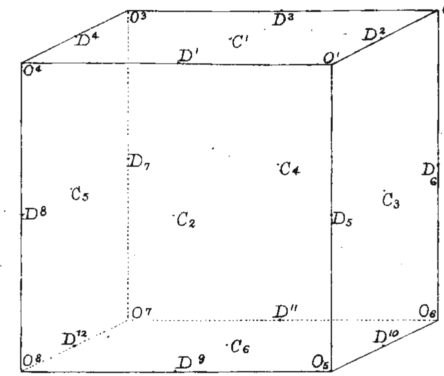

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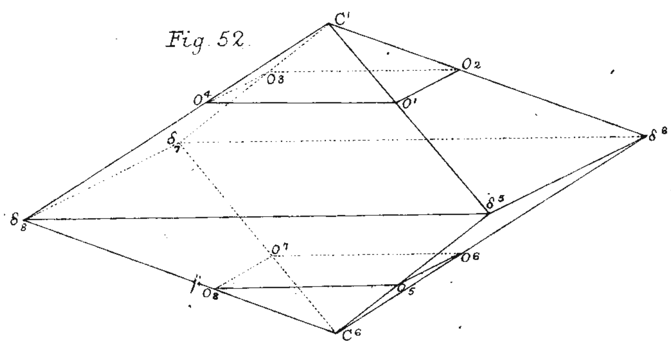

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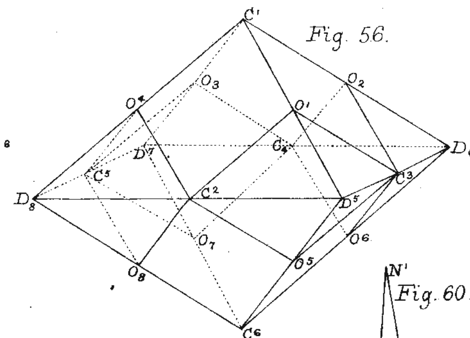

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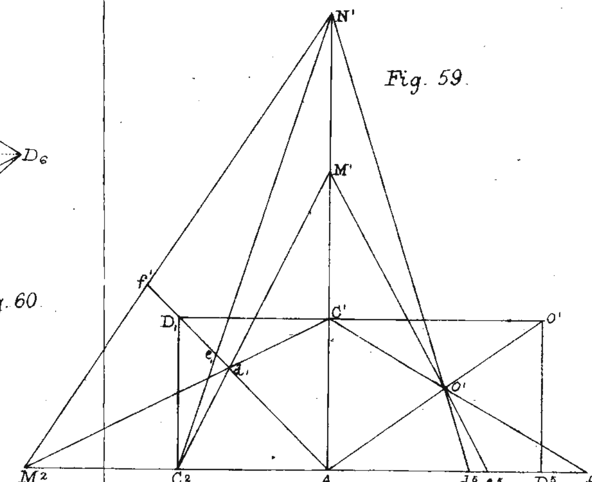
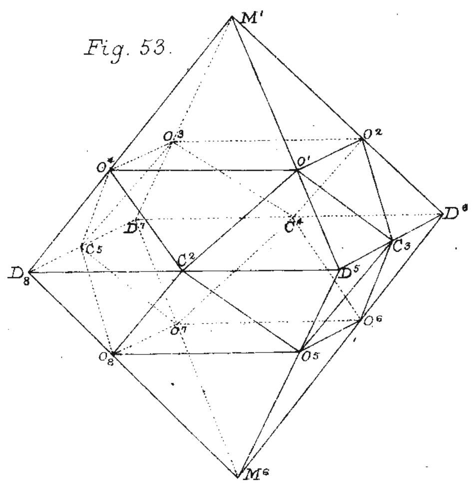


Fig. 53.



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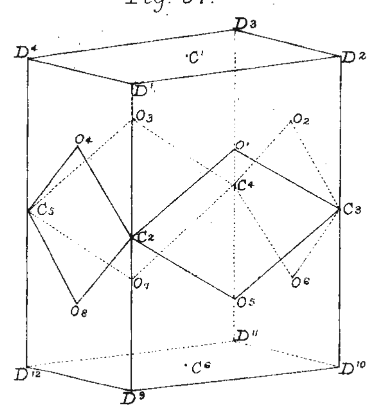


Fig. 60

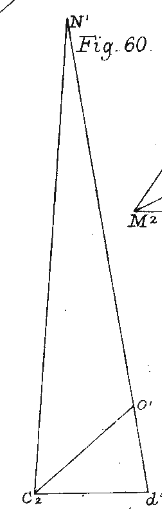
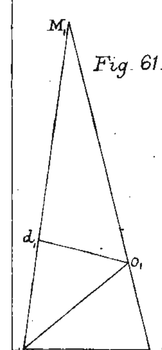


Fig. 61.



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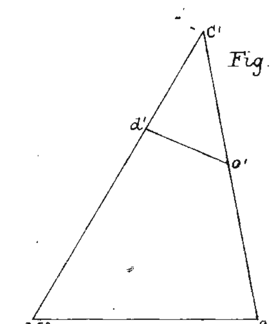

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Fig. 63.

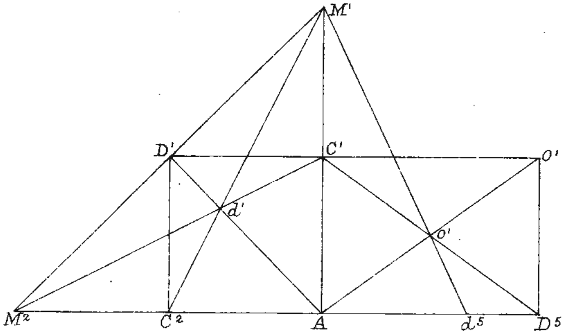
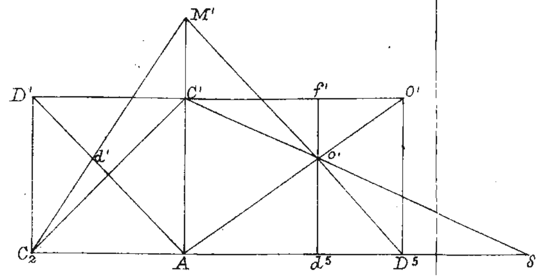


Fig. 69.



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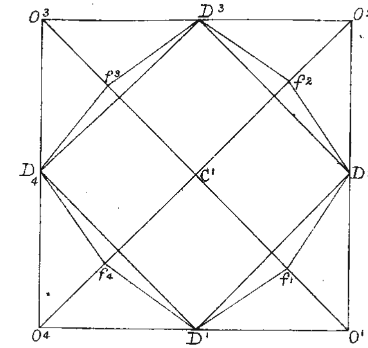

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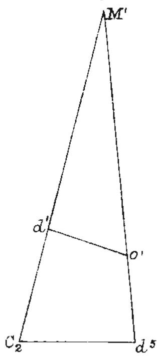

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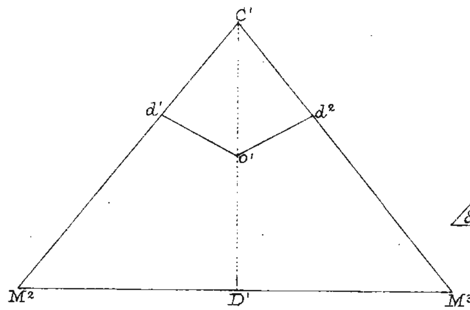

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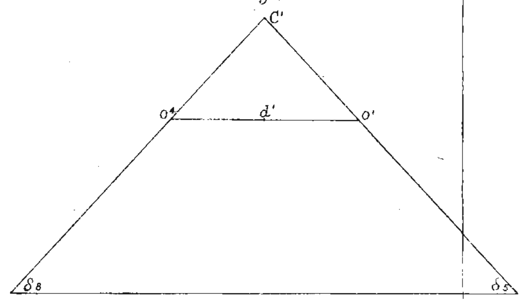

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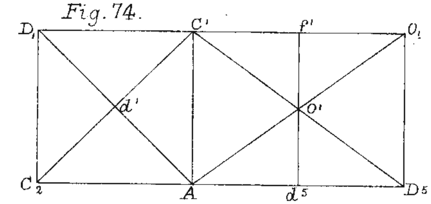

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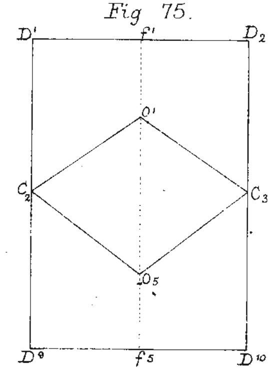
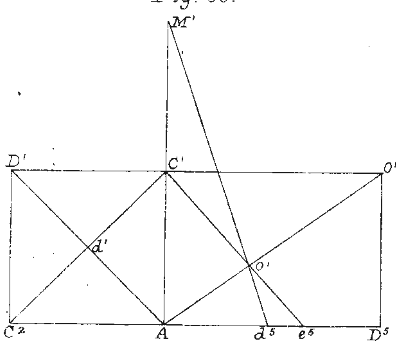


Fig. 66.



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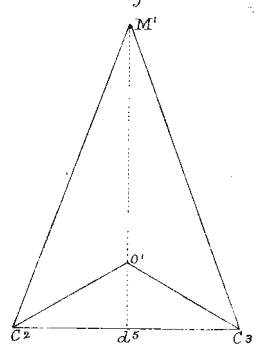

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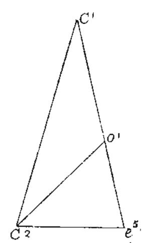

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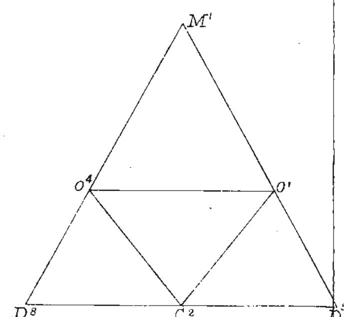

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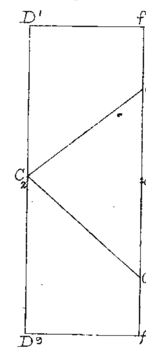

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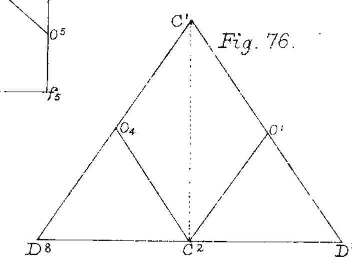

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Fig. 77. 1, $\frac{5}{4}$, $\frac{5}{3}$.

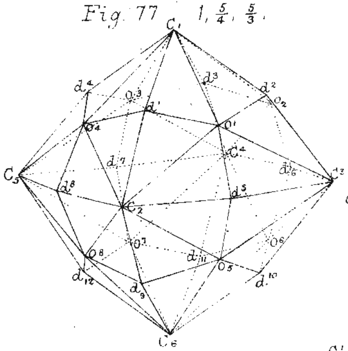


Fig. 78. 1. $\frac{3}{2}$. 3,

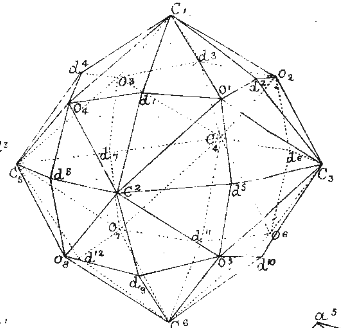


Fig. 79. 1. $\frac{4}{3}$, 4,

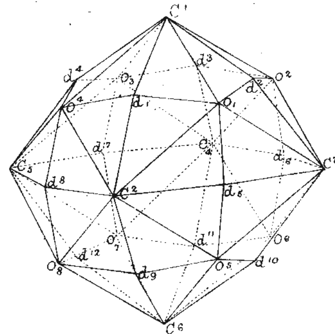


Fig. 80. 1. $\frac{4}{3}$, 4,

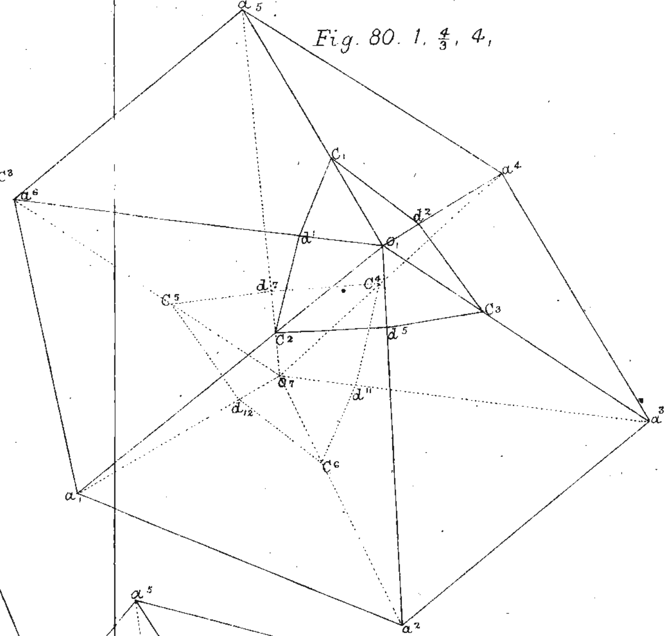


Fig. 81. 1, 4, 8.

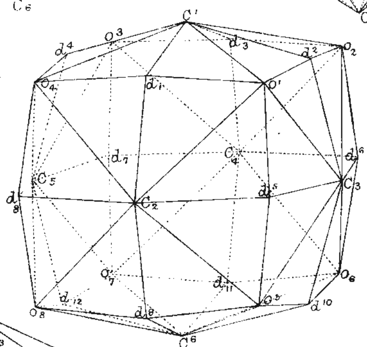


Fig. 82. 1, 4, 8,

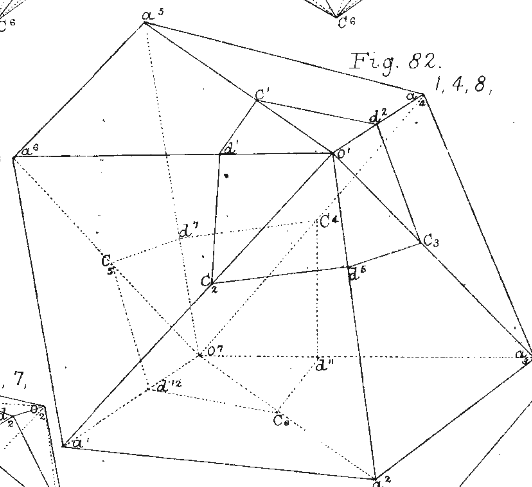


Fig. 83. 1. $\frac{5}{3}$, 5.

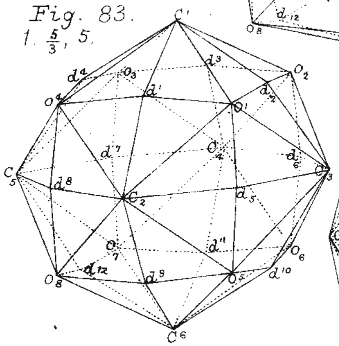


Fig. 84. 1, $\frac{7}{3}$, 7,

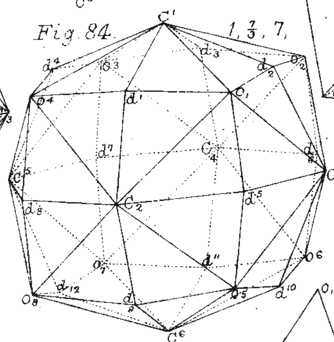


Fig. 85. 1, $\frac{3}{2}$, 3.

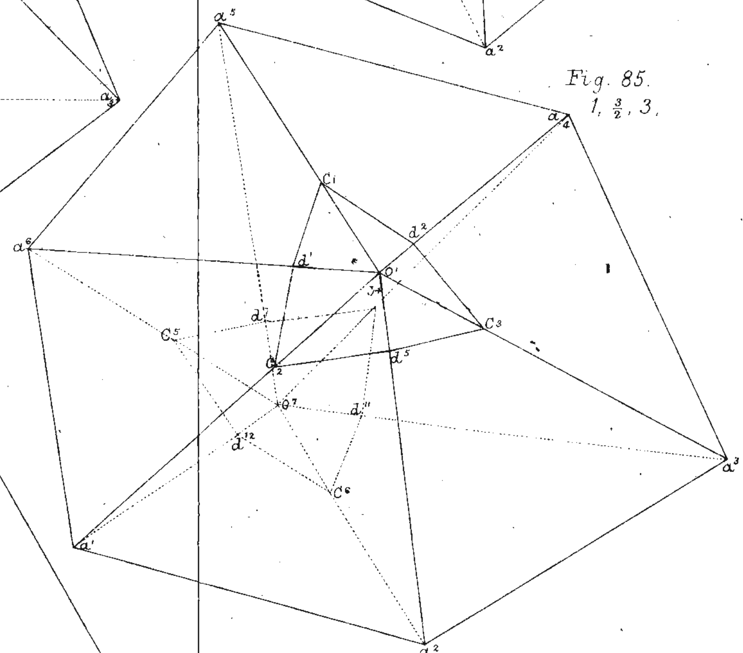


Fig. 86. 1, $\frac{4}{3}$. 4,

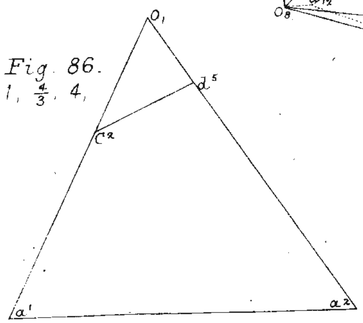


Fig. 87. 1, 4, 8.

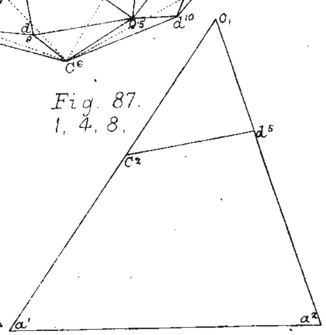
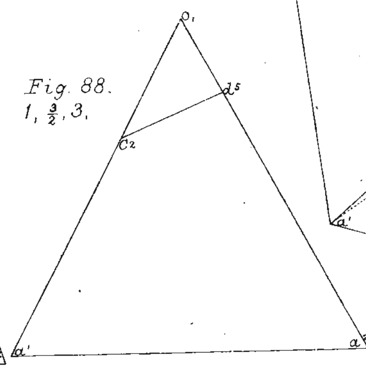


Fig. 88. 1, $\frac{3}{2}$, 3.



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